Lecture 6

Plane Waves

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ΔE_x
ΔE_y
ΔE_z

$\Delta E_x \rightarrow \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_z}{\partial z^2}$

$x^2 + 4x + 3$

$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial t} + C \rightarrow x = 2, 3, \ldots$

Review of the Previous Lecture

$E_x = E(x, y, z, t)$

$$\Delta \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\Delta \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$\vec{D} = \ldots$

Wave Equation

Skin Depth Equation

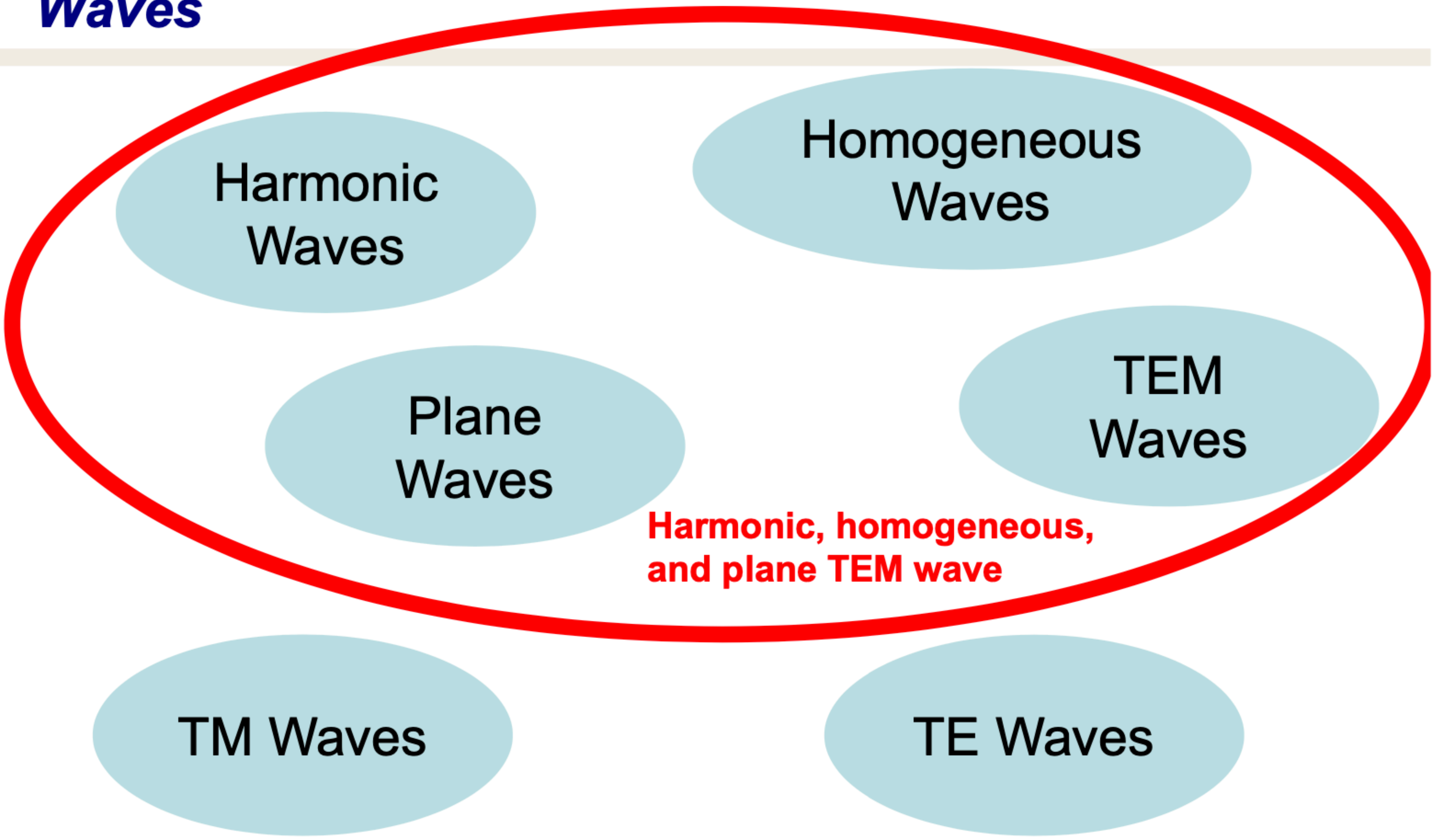
Good δ

$$\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}}$$

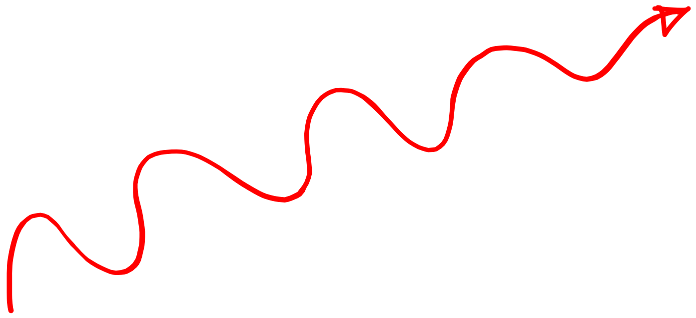
Bad δ

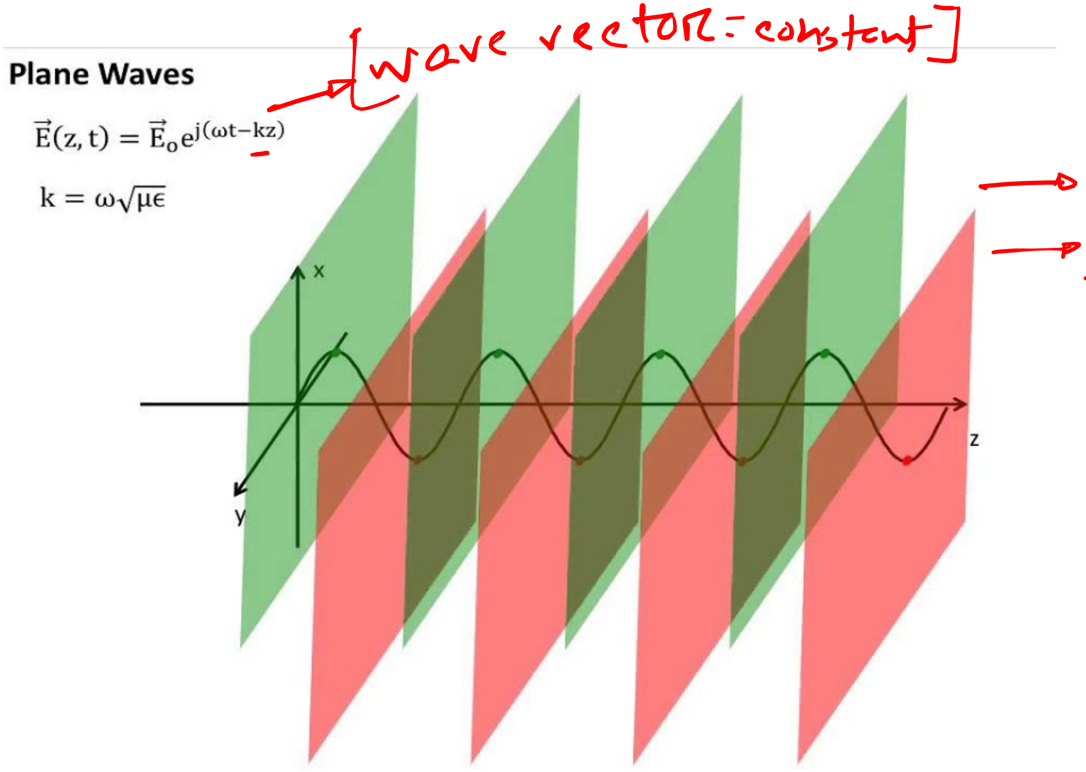
$$\delta_s = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

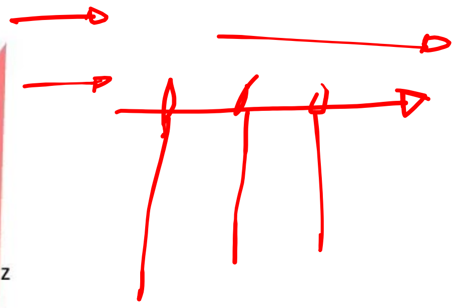
Waves


Harmonic Waves
Homogeneous Waves
Plane Waves
TEM Waves
Harmonic, homogeneous, and plane TEM wave
TM Waves
TE Waves

Plane Waves







- A plane wave is a special case of wave or field: a physical quantity whose value, at any moment, is constant over any plane that is perpendicular to a fixed direction in space.
- The phase surfaces (i.e., surfaces on which the phase of the wave is constant in any point of such a surface) are planes if the direction of wave propagation is unchanged.
- The wave vector **k** is the same at any position and at any time.

[Review>> Wave Vector: A wave vector is a vector indicating the direction of wave propagation and the phase delay per unit length.]

Homogeneous Waves

- The vector E0(x,y,z) is unchanged on a phase surface; this means that the wave’s intensity is the same in any point of such a surface.

Constant

$$\underline{\vec{E}}(x, y, z) = \boxed{\underline{\vec{E}}_0(x, y, z)}\, \mathrm{e}^{-j\underline{\vec{k}} \cdot \vec{r}}$$

Harmonic Waves

Wave Eqn Soln

$\tilde{\partial}/\tilde{\partial z}$

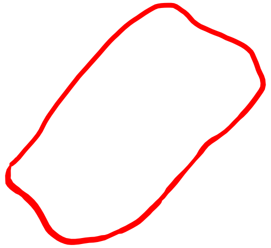
$\partial E_x / \partial z = 0$

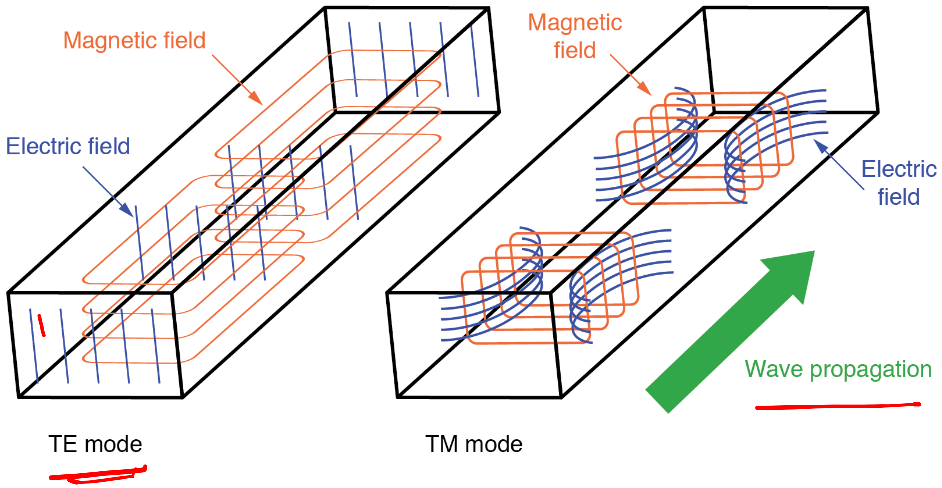
Not harmonic

$$\boxed{E_x = f(\omega t - \vec{k}\cdot\vec{r}) + g(\omega t + \vec{k}\cdot\vec{r})}$$

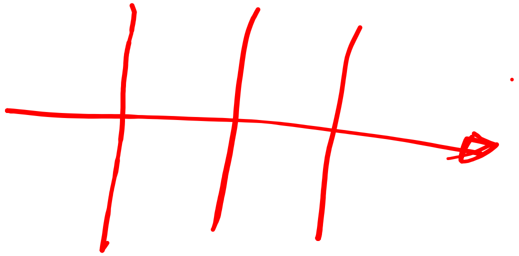
- A wave is called a harmonic wave if the general d'Alembert solutions f and g are harmonic functions (varying with angular frequency ω).
- [Review: **A harmonic function** is a twice continuously differentiable function]

Transverse Electric (TE) Waves

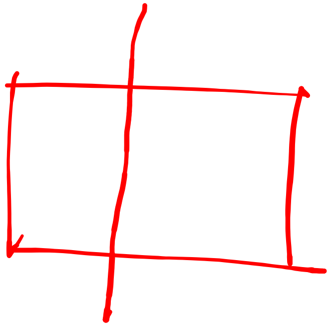




Magnetic flux lines appear as continuous loops
Electric flux lines appear with beginning and end points

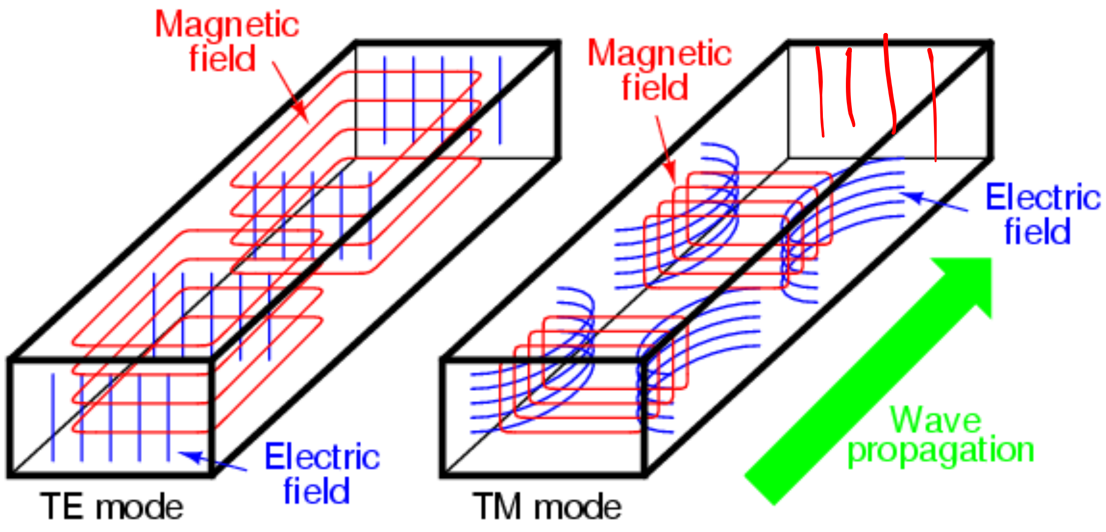


Electric



- The electric field vector does not have a field component directed in the direction of propagation, $\mathbf{E0}(x,y,z)\cdot\mathbf{k} = 0$.
- The magnetic field vector can have a field component directed in the direction of propagation.
- The electric field is parallel to the phase planes and perpendicular to the direction of propagation, $\mathbf{E0}(x,y,z) \perp \mathbf{k}$.

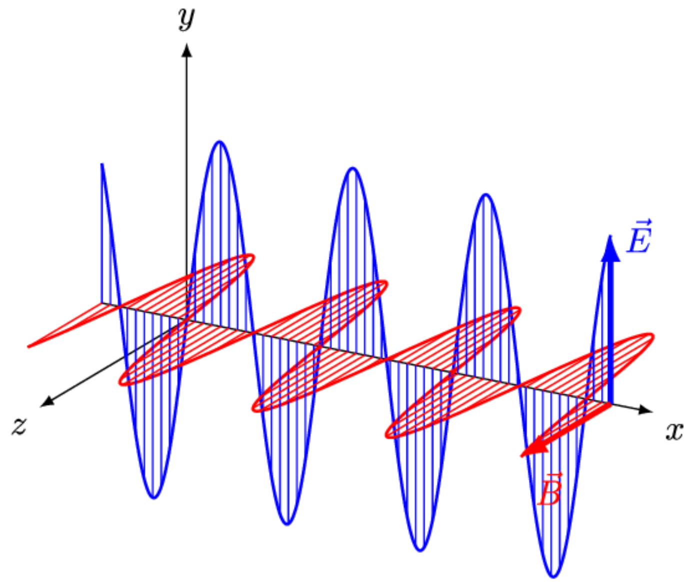
Transverse Magnetic (TM) Waves



Magnetic flux lines appear as continuous loops
Electric flux lines appear with beginning and end points

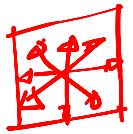
- The magnetic field vector does not have a field component directed in the direction of propagation, **H0**(x,y,z)·**k** = 0
- The electric field vector can have a field component directed in the direction of propagation.
- The magnetic field is parallel to the phase planes and perpendicular to the direction of propagation, **H0**(x,y,z) ⊥ k.

Transverse Electro-Magnetic (TEM) Waves

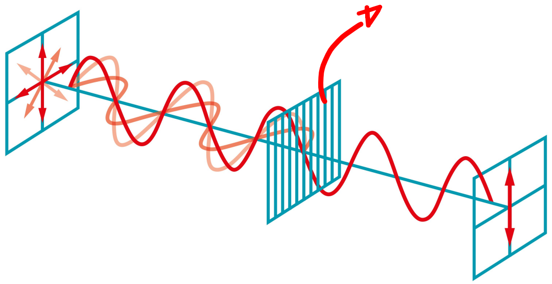


- A TEM wave has neither an electric nor a magnetic field component directed in the direction of propagation.
- Both the electric and the magnetic field vectors are parallel to the phase surface (i.e., perpendicular to the direction of propagation, **E0**(x, y, z) ⊥ **k**, **H0**(x, y, z) ⊥ **k**).

Linearly Polarized Waves



360°



- There exists a constant vector $\mathbf{e}$ such that $|\mathbf{E0}(x,y,z)\cdot\ \mathbf{e}| = |\mathbf{E0}(x,y,z)|$ holds for all times and all positions.
- The electric field vector does not change its orientation while propagating.
- For the indication of a linearly polarized wave, the direction of the electric field vector $\mathbf{E0}$ is used as the reference.
- **Example:** The waves of satellite TV are typically vertically or horizontally polarized TEM waves.

Plane Waves in Free Space

Assume a homogeneous and plane TE wave propagating harmonically in the z- direction. The wave should be linearly polarized in the y-direction. The medium is an insulator and free of charges.

$\vec{P} = \vec{E} \times \vec{H}$

$$\vec{E}(z,t) = \Re\left\{ \underline{\vec{E}}_0 \, \mathrm{e}^{\,j(\omega t - k_z z)} \right\}$$

$$\underline{E}_x = 0$$

$\boxed{2}$

$$\underline{E}_y = E_0 \, \mathrm{e}^{\,j(\omega t - k_z z)}$$ [because linearly polarized]

$$\underline{E}_z = 0$$

where E_0 is the magnitude of the electric vector field, $E_0 = |\underline{\vec{E}}_0|$. Due to the homogeneity of the wave the partial derivatives in x and y vanish:

Intensity constant

$$\frac{\partial \underline{E}_y}{\partial x} = 0, \qquad \frac{\partial \underline{E}_y}{\partial y} = 0$$

Plane Waves in Free Space

$\vec{\nabla} \times \vec{E}$

$$\text{curl}\,\underline{\vec{E}} = -j\omega\mu\underline{\vec{H}}$$ [Maxwell's Second Equation]

$$\text{curl}\underline{\vec{E}} = \begin{pmatrix} \frac{\partial \underline{E}_z}{\partial y} - \frac{\partial \underline{E}_y}{\partial z} \\ \frac{\partial \underline{E}_x}{\partial z} - \frac{\partial \underline{E}_z}{\partial x} \\ \frac{\partial \underline{E}_y}{\partial x} - \frac{\partial \underline{E}_x}{\partial y} \end{pmatrix} \checkmark$$

$$\underline{\vec{E}} = \begin{pmatrix} 0 \\ \underline{E}_y(z) \\ 0 \end{pmatrix}$$

$$\frac{\partial \underline{E}_y}{\partial x} = 0, \quad \frac{\partial \underline{E}_y}{\partial y} = 0$$

$$\begin{pmatrix} \cancel{\frac{\partial \underline{E}_z}{\partial y}} - \frac{\partial \underline{E}_y}{\partial z} \\ \cancel{\frac{\partial \underline{E}_x}{\partial z}} - \cancel{\frac{\partial \underline{E}_z}{\partial x}} \\ \cancel{\frac{\partial \underline{E}_y}{\partial x}} - \cancel{\frac{\partial \underline{E}_x}{\partial y}} \end{pmatrix} = -j\omega\mu \begin{pmatrix} \underline{H}_x \\ \cancel{\underline{H}_y} \\ \cancel{\underline{H}_z} \end{pmatrix} \checkmark$$

It must be a TEM Wave!

Plane Waves in Free Space

$\nabla = \vec{T} \vec{+} \vec{E}$

$$\frac{\partial \underline{E}_y}{\partial z} = -j\,k_z\,\underline{E}_y$$

$$\begin{pmatrix} \frac{\partial \cancel{\underline{E}_z}}{\partial y} - \frac{\partial \underline{E}_y}{\partial z} \\ \frac{\partial \cancel{\underline{E}_x}}{\partial z} - \frac{\partial \cancel{\underline{E}_z}}{\partial x} \\ \frac{\cancel{\partial \underline{E}_y}}{\cancel{\partial x}} - \frac{\partial \cancel{\underline{E}_x}}{\partial y} \end{pmatrix} = -j\omega\mu \begin{pmatrix} \underline{H}_x \\ \cancel{\underline{H}_y} \\ \cancel{\underline{H}_z} \end{pmatrix}$$

$$\underline{E}_y = E_0\,\mathrm{e}^{j(\omega t - k_z z)}$$

$$j\,k_z\,\underline{E}_y = -j\omega\mu\underline{H}_x$$

$$\underline{H}_x = -\frac{k_z}{\omega\mu}\underline{E}_y = -\frac{\omega\sqrt{\mu\varepsilon}}{\omega\mu}\underline{E}_y$$

$$= -\sqrt{\frac{\varepsilon}{\mu}}\,\underline{E}_y$$

$$= -\frac{1}{Z_F}\underline{E}_y$$

$Z_f = \sqrt{\mu / \epsilon}$

characteristic wave impedance

- The ratio of the *Electric (E)* and Magnetic (H) field components is seen to have units of impedance, known as the *characteristic wave impedance.*
- For planes waves the wave impedance is equal to the intrinsic impedance of the medium. In free-space the intrinsic impedance 377 Ohm.
- Note that the *Electric (E)* and *H* vectors are orthogonal to each other and orthogonal to the direction of propagation.

Plane Waves in Free Space – TEM Waves

$\vec{E} \perp \vec{H}$ and $\vec{E}$ and $\vec{H}$ are in phase! $\left|\vec{E}\right| = Z_F \left|\vec{H}\right|$

$$Z_F = \sqrt{\frac{\mu}{\varepsilon}} = Z_0 \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

$$\boxed{Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\,\Omega}$$

characteristic wave impedance in vaccum

$$\vec{H} = \frac{\vec{e}_z \times \vec{E}}{Z_F}$$

Orthogonal System

Plane Waves in Free Space – TEM Waves

The Poynting vector describes the flow of electromagnetic power and directed in the direction of propagation.

$$\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 0 & E_y & 0 \\ H_x & 0 & 0 \end{vmatrix}$$

$H_x E_y$

$-\vec{e}_z$

$$\begin{aligned}
\vec{S} &= \vec{E} \times \vec{H} \\
&= \begin{pmatrix} 0 \\ E_y \\ 0 \end{pmatrix} \times \begin{pmatrix} H_x \\ 0 \\ 0 \end{pmatrix} \\
&= -E_y H_x \vec{e}_z \quad ? \\
&= \frac{1}{Z_F} E_y^2 \; \vec{e}_z \quad ✓
\end{aligned}$$

$$\underline{H_x} = -\frac{1}{Z_F} \underline{E_y}$$

From previous equation

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{Z_F} E_y^2 \; \vec{e}_z$$

4.1 Problem 1

A plane wave propagates in a lossless medium. The electric vector field is given as:

$$\underline{\vec{E}} = \underline{E}_0 \mathrm{e}^{-jkz}\, \vec{e}_x$$

a) Relate k, ω, μ, and ε.

b) Give an expression for the wave impedance and the magnetic vector field $\underline{\vec{H}}$.

c) Using eq. (4.1.1), give an expression for the wavelength λ.

d) Express and discuss the phase velocity v_{ph} of the wave.

e) Determine the phase velocity v_{ph} of an electromagnetic wave in free space.

a) wave number $k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}$

ε_0 : permittivity of free space

$= 8.8542 \cdot 10^{-12} \frac{As}{Vm}$

μ_0: permeability of free space

$= 4\pi\, 10^{-7} \frac{Vs}{Am}$

4.1 Problem 1

$e^{j\theta}$

A plane wave propagates in a lossless medium. The electric vector field is given as:

$$\vec{E} = E_0 e^{-jkz}\,\vec{e}_x \qquad e^{j\omega t}$$

$\vec{E}$ $\vec{E}$

a) Relate k, ω, μ, and ε.

b) Give an expression for the wave impedance and the magnetic vector field $\vec{H}$.

c) Using eq. (4.1.1), give an expression for the wavelength λ.

d) Express and discuss the phase velocity v_{ph} of the wave. ① $\frac{E_0}{Z_F}$ ② $\vec{e}_y$

e) Determine the phase velocity v_{ph} of an electromagnetic wave in free space.

$$\vec{E}(x,t) = \vec{E}\, e^{j\omega t} \qquad \vec{E} = E_0 \cos(\omega t - kz)\,\hat{e}_x$$

- The ratio of the *Electric (E)* and Magnetic (H) field components is seen to have units of impedance, known as the *characteristic wave impedance.*

$\frac{2\pi}{T}$

k = wave number

($\vec{k}$ = wave vector) → wave propagation

b) $Z = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}$

magnetic vector field $\vec{H}$: $\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's Law)

$= -j\omega\mu\vec{H}$

$\Rightarrow \vec{H} = -\frac{1}{j\omega\mu}\,\text{curl}\,\vec{E}$

here: $\vec{E} = E_x\vec{e}_x$ $\quad \text{curl}\,\vec{E} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ \frac{\partial E_x}{\partial z} \\ -\frac{\partial E_x}{\partial y} \end{pmatrix}$

here: $\vec{E}(x,t) = E_0 e^{j(\omega t - kz)}\vec{e}_x$

$$\frac{\partial E_x}{\partial y} = 0;\quad \frac{\partial E_x}{\partial z} = -jk\,E_0 e^{j(\omega t - kz)}$$

$$\Rightarrow \vec{H}(x,t) = \begin{pmatrix} 0 \\ -jkE_0 e^{j(\omega t - kz)} \\ 0 \end{pmatrix} \cdot \left(-\frac{1}{j\omega\mu}\right) = \begin{pmatrix} 0 \\ \frac{k}{\omega\mu}E_0 e^{j(\omega t - kz)} \\ 0 \end{pmatrix}$$

$$\frac{|\vec{E}|}{|\vec{H}|} = \frac{E_0}{\frac{k}{\omega\mu}E_0} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = Z$$

$$\vec{H} = \begin{pmatrix} 0 \\ \frac{k}{\omega\mu}E_0 e^{-jkz} \\ 0 \end{pmatrix}$$

4.1 Problem 1

A plane wave propagates in a lossless medium. The electric vector field is given as:

$$\underline{\vec{E}} = E_0 e^{-jkz}\,\vec{e}_x$$

$\cos(\omega t - kz)$

a) Relate k, ω, μ, and ε.

b) Give an expression for the wave impedance and the magnetic vector field $\underline{\vec{H}}$.

c) Using eq. (4.1.1), give an expression for the wavelength λ.

d) Express and discuss the phase velocity v_{ph} of the wave.

e) Determine the phase velocity v_{ph} of an electromagnetic wave in free space.

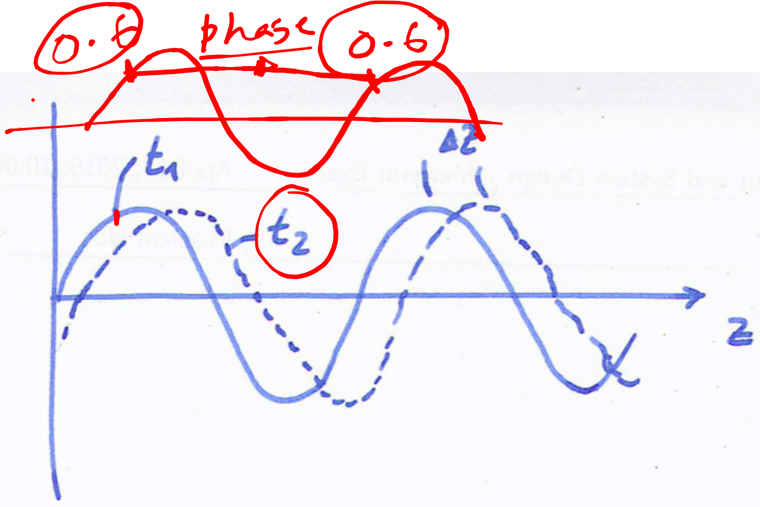
c) $\cancel{\omega t} - k\cancel{z} = \cancel{\omega t} - k(\cancel{z} + \lambda) + 2\pi$

$0 = -k\lambda + 2\pi$

$k = \dfrac{2\pi}{\lambda}$; $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \dfrac{1}{f\sqrt{\mu\varepsilon}}$

wave vector

d)



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t_1, t_2 same phase

$t_2 = t_1 + \Delta t$

→ Phase at t_1 : $\omega t_1 - k z_0$

→ Phase at t_2 : $\omega t_2 - k(z_0 + \Delta z) = \omega(t_1 + \Delta t) - k(z_0 + \Delta z)$

→ $\omega(\cancel{t_1} + \Delta t) - k(\cancel{z_0} + \Delta z) = \cancel{\omega t_1} - \cancel{k z_0}$

→ $\omega \Delta t - k \Delta z = 0$

$$\frac{\Delta z}{\Delta t} = v_{ph} = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

→ Phase সমান

4.1 Problem 1

A plane wave propagates in a lossless medium. The electric vector field is given as:

$$\underline{\vec{E}} = E_0 \mathrm{e}^{-jkz}\, \vec{e}_x$$

a) Relate k, ω, μ, and ε.

b) Give an expression for the wave impedance and the magnetic vector field $\underline{\vec{H}}$.

c) Using eq. (4.1.1), give an expression for the wavelength λ.

d) Express and discuss the phase velocity v_{ph} of the wave.

e) Determine the phase velocity v_{ph} of an electromagnetic wave in free space.

$$v_{ph} = \frac{c_0}{\sqrt{\mu_r \varepsilon_r}}$$

e) Free space : $\varepsilon = \varepsilon_0 = 8.8542 \cdot 10^{-12} \frac{As}{Vm}$

$\mu = \mu_0 = 4\pi 10^{-7} \frac{Vs}{Am}$

$v_{ph} = c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99792458 \frac{m}{s}$

$\approx 300\,000 \frac{km}{s}$

$\approx 300 \frac{km}{ms}$

$\approx 300 \frac{m}{\mu s}$

$\approx 30\,cm/ns$

$3 \times 10^8\ ms^{-1}$

Lecture 7

Plane Wave Maths

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4.2 Problem 2

A plane wave travels in the $+z$ direction in a dielectric lossless medium with a relative permittivity of $\varepsilon_r = 9$, at a frequency of $300\,\mathrm{MHz}$ and with an electric field amplitude of $100\,\mathrm{V/m}$.

a) Write the complete time-domain expressions for the vector fields $\vec{E}$ and $\vec{H}$.

b) Calculate the average power density of the wave.

Now, assume the dielectric material has a conductivity $\sigma = 10\,\mathrm{S\,m^{-1}}$.

c) What are the wave impedance and wave number of the wave?

d) Determine the average power density of the wave.

$+x \rightarrow$

$E_{y/z} = E_0 e^{j(\omega t - kx)}$

$$E_y = E_0\, \mathrm{e}^{j(\omega t - k_z z)}$$

$\frac{E}{H} = Z_F$

$$Z_F = \sqrt{\frac{\mu}{\varepsilon}}$$

$H_x = \frac{E_y}{Z_F}$

$$Z_0 = \sqrt{\mu_0/\varepsilon_0}$$

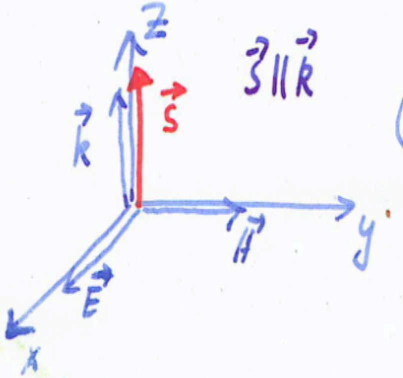
a) $\vec{E}(\omega t) = E_0 \cos(\omega t - kz)\,\vec{e}_x$; $E_0 = 100\,\frac{V}{m}$

(or $\vec{E}(\omega t) = E_0 \cos(\omega t - kz)\,\vec{e}_y$)

(or a mixture of both)

with $E_0 = 100\,\frac{V}{m}$; $\omega = 2\pi f = 2\pi \cdot 300 \cdot 10^6\,\frac{1}{s}$

$\hat{e}_z \times \hat{e}_x$
$\hat{e}_y$



$\vec{S} \parallel \vec{k}$

$(\vec{E} \times \vec{H}) \parallel \vec{k}$; $\vec{H} \parallel (\vec{k} \times \vec{E})$; here: $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$

$\vec{H} = \vec{e}_z \times \vec{E} \cdot \frac{1}{Z}$

$Z = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_r}} = Z_0 \cdot \sqrt{\frac{1}{\varepsilon_r}} = \frac{1}{3} Z_0$ with $Z_0 \approx 120\pi\,\Omega$

$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix} \cdot \frac{1}{Z} = \begin{pmatrix} 0 \\ \frac{E_x}{Z} \\ 0 \end{pmatrix} = \frac{E_0}{Z} \cos(\omega t - kz)\,\vec{e}_y$

4.2 Problem 2

A plane wave travels in the $+z$ direction in a dielectric lossless medium with a relative permittivity of $\varepsilon_r = 9$, at a frequency of $300\,\mathrm{MHz}$ and with an electric field amplitude of $100\,\mathrm{V/m}$.

a) Write the complete time-domain expressions for the vector fields $\vec{E}$ and $\vec{H}$.

b) Calculate the average power density of the wave.

Now, assume the dielectric material has a conductivity $\sigma = 10\,\mathrm{S\,m^{-1}}$.

c) What are the wave impedance and wave number of the wave?

d) Determine the average power density of the wave.

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{Z_F} E_y^2 \, \vec{e}_z$$

$$\vec{e}_x \quad \vec{e}_y \quad \vec{e}_z$$
$$E_x \to H_y \quad 0 \quad 0$$
$$0 \quad 0 \quad 0$$

$$\vec{e}_z (E_x H_y)$$

b)
$$\vec{S} = \vec{E} \times \vec{H} = \begin{pmatrix} E_0 \cos(\omega t - kz) \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ \frac{E_0}{Z} \cos(\omega t - kz) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{E_0^2}{Z} \cos^2(\omega t - kz) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{E_0^2}{2Z}\left(1 + \cos(2\omega t - 2kz)\right) \end{pmatrix}$$

Average power density

$$\overline{\vec{S}} = \frac{1}{T} \int_0^T \vec{S}(\omega t)\, dt = \frac{E_0^2}{2Z} \vec{e}_z = 39.79 \frac{W}{m^2} \vec{e}_z$$

$E_0 = 100\,V/m, \; Z = 40\pi$

4.2 Problem 2

A plane wave travels in the $+z$ direction in a dielectric lossless medium with a relative permittivity of $\varepsilon_r = 9$, at a frequency of $300\,\mathrm{MHz}$ and with an electric field amplitude of $100\,\mathrm{V/m}$.

a) Write the complete time-domain expressions for the vector fields $\vec{E}$ and $\vec{H}$.

b) Calculate the average power density of the wave.

Now, assume the dielectric material has a conductivity $\sigma = 10\,\mathrm{S\,m^{-1}}$. → lossy

c) What are the wave impedance and wave number of the wave?

d) Determine the average power density of the wave.

$$K = \omega\sqrt{\mu\underline{\varepsilon}} = \frac{2\pi}{\lambda}$$

$$Z_F = \sqrt{\frac{\mu}{\underline{\varepsilon}}} \qquad \underline{\varepsilon} = \varepsilon\left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right)$$

$$\tan\delta = \frac{\sigma}{\omega\,\varepsilon} \qquad \text{with} \quad \varepsilon = \varepsilon_0\,\varepsilon_r$$

4.2c) $\sigma = 10\,\frac{S}{m}$ (very lossy!) (salt water) (16)

impedance:

$$\underline{Z} = \sqrt{\frac{\mu}{\underline{\varepsilon}}} \quad \text{with } \mu = \mu_0 \text{ and}$$

$$\underline{\varepsilon} = \varepsilon_0\varepsilon_r(1 - j\tan\delta)$$

$$\tan\delta = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r}$$

$$= \frac{10\,\frac{S}{m}}{2\pi\cdot 300\cdot 10^6\,\frac{1}{s}\cdot 8.854\cdot 10^{-12}\,\frac{As}{Vm}\cdot 9}$$

$$= 66.57 \quad \text{(no unit!)}$$

$$\underline{Z} = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r(1-j\tan\delta)}} = \frac{Z_0}{\sqrt{\varepsilon_r(1-j\tan\delta)}}$$

$$\text{with } Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi\,\Omega$$

Complex

$$1 - j\tan\delta = 1 - j66.57 \approx 66.58\,e^{-j89.14^\circ}$$

$$\frac{1}{\sqrt{1-j\tan\delta}} = \frac{1}{\sqrt{66.58\,e^{-j89.14^\circ}}} \approx \frac{1}{8.16}\,e^{+j44.57^\circ}$$

$$\left(\frac{1}{\sqrt{e^{-j\varphi}}} = e^{+j\varphi/2}\right)$$

$$\Rightarrow \underline{Z} = \frac{Z_0}{\sqrt{\varepsilon_r}}\cdot\frac{1}{8.16}\,e^{j44.57^\circ} \approx 10.97\,\Omega + j\,10.81\,\Omega$$

wave number:

$$\underline{k} = \omega\sqrt{\mu\underline{\varepsilon}} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}\sqrt{1-j\tan\delta}$$

$$= \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}\cdot 8.16\,e^{-j44.57^\circ}$$

$$\approx (109.6 - j108)\,\frac{1}{m}$$

$$= k' - jk''$$

Complex

4.2 Problem 2

A plane wave travels in the $+z$ direction in a dielectric lossless medium with a relative permittivity of $\varepsilon_r = 9$, at a frequency of $300\,\mathrm{MHz}$ and with an electric field amplitude of $100\,\mathrm{V/m}$.

a) Write the complete time-domain expressions for the vector fields $\vec{E}$ and $\vec{H}$.

b) Calculate the average power density of the wave.

Now, assume the dielectric material has a conductivity $\sigma = 10\,\mathrm{S\,m^{-1}}$.

c) What are the wave impedance and wave number of the wave?

d) Determine the average power density of the wave.

$$\vec{S} = \frac{1}{T}\int_0^T \vec{S}(t)\,dt = \frac{1}{2}\Re\{\vec{E}\times\vec{H}^*\}$$

$$\underline{Z} = \frac{Z_0}{\sqrt{\varepsilon_r}}\cdot\frac{1}{8.16}e^{j44.57^\circ} \approx 10.97\,\Omega + j\,10.81\,\Omega$$

$E_x = E_0\cos(\omega t - kx)$

$H_y = H_0\cos(\omega t - kx + \Phi)$

$a+ib$

$a-ib$

4.2 d)

$$\vec{E}(\omega t) = \mathrm{Re}\{\vec{\underline{E}}_0\, e^{j\omega t}\}$$

$$\text{with } \vec{\underline{E}}_0 = 100\tfrac{V}{m}\cdot e^{-jkz}\cdot\vec{e}_x$$

$$= 100\tfrac{V}{m}\, e^{-j(k'-jk'')z}\,\vec{e}_x$$

$$= 100\tfrac{V}{m}\cdot\underbrace{e^{-k''z}}_{\text{damping}}\cdot\underbrace{e^{-jk'z}}_{\text{phasor}}\,\vec{e}_x$$

$$\vec{H}(\omega t) = \mathrm{Re}\{\vec{\underline{H}}_0\, e^{j\omega t}\}$$

$$\text{with } \vec{\underline{H}}_0 = \frac{E_0}{\underline{Z}}e^{-k''z}\cdot e^{-jk'z}\,\vec{e}_y$$

$$\Rightarrow \vec{\overline{S}} = \tfrac{1}{2}\mathrm{Re}\{\vec{\underline{E}}\times\vec{\underline{H}}^*\} = \mathrm{Re}\{\vec{\underline{S}}\}$$ (mean active power flow)

$$\underline{Z} = |\underline{Z}|e^{j\varphi_z} \quad (= 10.97\,\Omega + j10.81\,\Omega)$$

$$\Rightarrow \vec{\underline{H}}_0 = \frac{E_0}{|\underline{Z}|}\cdot e^{-k''z}\cdot e^{-j(k'z+\varphi_z)}\cdot\vec{e}_y$$ (phase)

$$\vec{\underline{S}} = \tfrac{1}{2}\vec{\underline{E}}_0\times\vec{\underline{H}}_0^* = \frac{E_0^2}{2|\underline{Z}|}\cdot e^{-2k''z}\cdot e^{+j\varphi_z}\cdot\vec{e}_z$$

$$\Rightarrow \vec{\overline{S}} = \tfrac{1}{2}\mathrm{Re}\{\vec{\underline{E}}\times\vec{\underline{H}}^*\} = \frac{E_0^2}{2|\underline{Z}|}e^{-2k''z}\cos(\varphi_z)\cdot\vec{e}_z$$

$$= \frac{(100\frac{V}{m})^2}{2\sqrt{10.97^2+10.81^2}\,\Omega}\cdot\cos(44.57^\circ)\,e^{-2k''z}\cdot\vec{e}_z$$

$$= 231.3\,e^{-2k''z}\,\frac{W}{m^2}\cdot\vec{e}_z$$