Lecture 5

Wave Equations, Solutions and Plane Waves

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Electromagnetic Waves

$$\text{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{div}\,\vec{D} = \varrho$$

$$\text{div}\,\vec{B} = 0$$

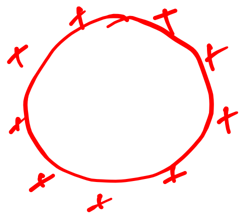
Electromagnetic Waves

Wave Equation:

$\vec{E} = \frac{1}{4\pi\epsilon_0} \times \frac{q}{r^2}$

$\vec{D} = \frac{q}{4\pi r^2}$

$\vec{D} = \epsilon \vec{E}$



From Maxwell's Eqn:

$$\text{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{--- ①}$$

$\sigma \vec{E}$, $\epsilon\vec{E}$

$$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (i)}$$

$$\text{curl curl}\,\vec{H} = \text{grad div}\,\vec{H} - \Delta \vec{H}. \quad \text{--- (iii)}$$

$$\text{div}\,\vec{B} = 0 \quad \text{--- (ii)}$$

$\mu\,\text{div}\,\vec{H} = 0$

$$\text{curl curl}\,\vec{H} = \sigma\,\text{curl}\,\vec{E} + \varepsilon \frac{\partial}{\partial t}\left(\text{curl}\,\vec{E}\right)$$

$$\text{curl curl}\,\vec{H} = -\mu\sigma\frac{\partial \vec{H}}{\partial t} + \varepsilon\frac{\partial}{\partial t}\left((-\mu)\frac{\partial \vec{H}}{\partial t}\right)$$

[① থেকে Value plugin করে]

$$= -\mu\sigma\frac{\partial \vec{H}}{\partial t} - \mu\varepsilon\frac{\partial^2 \vec{H}}{\partial t^2}$$

$$-\Delta\vec{H} \quad = \quad ''$$

$$\boxed{\Delta\vec{H} = \mu\sigma\frac{\partial \vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2 \vec{H}}{\partial t^2}}$$

$\nabla = \partial/\partial x\, \hat{i} + \text{-----}$

$\Delta = \partial^2/\partial x^2\, \hat{i} + \text{-----}$

Electromagnetic Waves

$① \times \mu \to \operatorname{curl} \vec{B} = \mu\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$

Wave Equation:

$\frac{\partial}{\partial x}\phi_x \hat{i} + \frac{\partial}{\partial y}\phi_y \hat{j} +$

$$\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\operatorname{curl} \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{--}①$$

$$\operatorname{div} \vec{D} = \varrho \quad \text{--}②$$

$$\begin{aligned}
\operatorname{curl}\operatorname{curl} \vec{E} &= -\mu \frac{\partial}{\partial t}\left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right) \\
&= -\mu\sigma \frac{\partial \vec{E}}{\partial t} + \varepsilon \frac{\partial}{\partial t}\left((-\mu)\frac{\partial \vec{E}}{\partial t}\right) \\
&= -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}
\end{aligned}$$

Used in here

$$\begin{aligned}
\operatorname{curl}\operatorname{curl} \vec{E} &= \operatorname{grad}\operatorname{div} \vec{E} - \Delta \vec{E} \\
&= \operatorname{grad} \frac{\varrho}{\varepsilon} - \Delta \vec{E}
\end{aligned}$$

$\operatorname{div}(\varepsilon \vec{E}) = \rho$

$\boxed{\operatorname{div} \vec{E} = \frac{\rho}{\varepsilon}}$

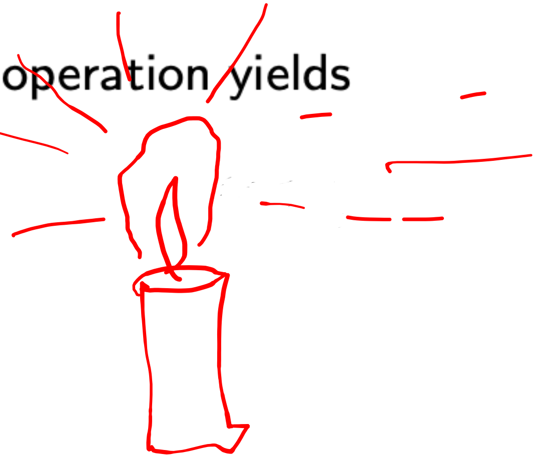
Used in here

Because of the homogeneity of the medium and of the charge density ϱ, the grad operation yields zero

charge equal

$$\operatorname{grad} \frac{\varrho}{\varepsilon} \equiv \vec{0}$$

$$\boxed{\Delta \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$



Electromagnetic Waves

Laplace Operator:

$$\Delta\vec{H} = \begin{pmatrix} \Delta H_x \\ \Delta H_y \\ \Delta H_z \end{pmatrix}$$

$$\Delta\vec{H} = \begin{pmatrix} \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} \\ \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} \\ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} \end{pmatrix}$$

$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

$\vec{\nabla}\cdot\vec{H} = \frac{\partial}{\partial x}H_x + \partial/\partial y\, H_y + \partial/\partial z\, H_z$

$\vec{\nabla}\times\vec{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix}$

Electromagnetic Waves

Wave Equation in Complex Notation:

$\frac{\partial}{\partial t}\left(e^{j\omega t}\right)$

$= j\omega e^{j\omega t}$

$\frac{\partial^2}{\partial t^2} = -\omega^2$

Let us now consider **harmonic fields** and use complex notation with the general time dependence of $e^{j\omega t}$. Then the rate of change equals a multiplication with $j\omega$ meaning that the second derivative wrt. time is

$\vec{H} = H_0 e^{j\omega t}$

$$\boxed{\Delta\vec{H} = \mu\sigma\frac{\partial \vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2 \vec{H}}{\partial t^2}} \qquad \frac{\partial \ldots}{\partial t} \equiv -\omega^2 \qquad \boxed{\Delta\vec{E} = \mu\sigma\frac{\partial \vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2}}$$

$$\begin{aligned}\Delta\underline{\vec{H}} &= \left(-\omega^2\mu\varepsilon + j\omega\mu\sigma\right)\underline{\vec{H}} \\ &= -\omega^2\mu\varepsilon\left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right)\underline{\vec{H}}\end{aligned}$$

$$\begin{aligned}\Delta\underline{\vec{E}} &= \left(-\omega^2\mu\varepsilon + j\omega\mu\sigma\right)\underline{\vec{E}} \\ &= -\omega^2\mu\varepsilon\left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right)\underline{\vec{E}}\end{aligned}$$

complex

By comparing these equations with the case of an insulator $(\sigma = 0)$,

$$\Delta\underline{\vec{H}} = -\omega^2\mu\varepsilon\,\underline{\vec{H}}$$

$$\Delta\underline{\vec{E}} = -\omega^2\mu\varepsilon\,\underline{\vec{E}}$$

Electromagnetic Waves

$\underline{\varepsilon} = \varepsilon\left(1 - j\frac{\sigma}{\varepsilon\omega}\right)$

Wave Equation in Complex Notation:

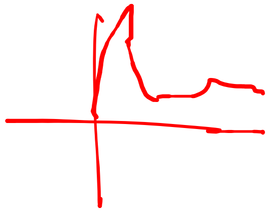
[∴ insulators]

$\varepsilon \rightarrow$ complex permittivity

$$\Delta \underline{\vec{H}} = -\omega^2 \mu \varepsilon \left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right) \underline{\vec{H}}$$

$$\Delta \underline{\vec{H}} = -\omega^2 \mu \underline{\varepsilon}\, \underline{\vec{H}}$$

$$\Delta \underline{\vec{E}} = -\omega^2 \mu \underline{\varepsilon}\, \underline{\vec{E}}$$



with

$$\underline{\varepsilon} = \varepsilon\left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right)$$

$$= \varepsilon\left(1 - j\frac{1}{\tau_{\text{relax}}\,\omega}\right)$$

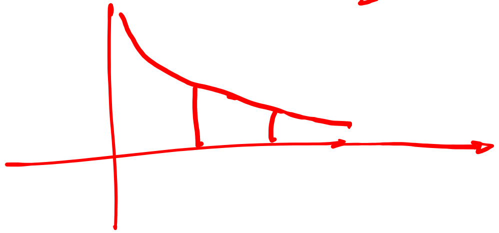
$$= \varepsilon \cdot (1 - j\ \tan\delta)$$

$\tau_{\text{relax}} \rightarrow$ relaxation time

$\tan\delta = \frac{\sigma}{\omega\,\varepsilon}$ with $\varepsilon = \varepsilon_0\,\varepsilon_r$

Loss Tangent (Dissipation Factor)

$\tau = \tau_0 e^{-t/\tau}$



Electromagnetic Waves

Taconic High Performance Materials - Typical Values													
Part #	Composition	Dk	Df	Volume Resistivity Mohm/cm	Surface Resistivity Mohm	Flex Strength kpsi		Moisture Absorption %	Thermal Conductivity W/M*K	CTE * ppm/°C			Peel Strength lbs/in Very Low Profile Cu
		$=\varepsilon_r$	$=\tan\delta$			MD	CD			x	y	z	CV1
TLY-5A	PTFE - Glass	2.17[a]	0.0009[a]	10^{10}	10^{9}	14	13	<0.02	0.22	20	20	280	12
TLP-5	PTFE - Glass	2.20[a]	0.0009[a]	10^{7}	10^{7}	>12	>10	<0.02	0.22	20	20	280	10
TLY-5	PTFE - Glass	2.20[a]	0.0009[a]	10^{10}	10^{8}	14	13	<0.02	0.22	20	20	280	12
TLY-3	PTFE - Glass	2.33[a]	0.0012[a]	10^{10}	10^{8}	14	13	<0.02	0.22	20	20	280	12
TLX-0, 9, 8, 7, 6	PTFE - Glass	2.45 - 2.65[a]	0.0015 - 0.0021[c]	10^{10}	10^{8}	>23	>17	<0.02	0.22	9	12	140	12
TLA-6	PTFE - Glass	2.65	0.0017	10^{12}	10^{14}	13	12	0.02	0.19	9	12	140	>10
TSM-26, 30	PTFE Ceramic - Glass	2.60[c], 3.00[c]	0.0014[c], 0.0013[c]	10^{8}	10^{7}	>7	>6	0.03	0.27	23	28	78	8
*fast*Film™27 (FF-27)[Ŧ]	PTFE Ceramic - Film	2.70	0.0012	10^{8}	10^{8}	4	4	0.03	0.34	29	28	112	16
TLC-27, 30, 32, 338, 35	PTFE - Glass	2.75 - 3.50[a]	0.0023 - 0.0037[c]	10^{8}	10^{8}	24	19	<0.02	0.24	9	12	70	12
TSM-DS	PTFE Ceramic - Glass	2.85[a]	0.0010[a]	10^{8}	10^{8}	15	8	0.04	0.45	11	18	57	8
TLE-95	PTFE - Glass	2.95[a]	0.0026[a]	10^{9}	10^{7}	>24	>18	<0.02	0.28[e]	9	12	70	12
RF-30	PTFE Ceramic - Glass	3.00[b]	0.0014[b]	10^{9}	10^{8}	20	16	0.02	0.20	11	21	208	10
TSM-DS3	PTFE Ceramic - Glass	3.00[a]	0.0011[a]	10^{8}	10^{6}	12	8	0.07	0.65	10	16	23	8
TRF-43, 45	PTFE Ceramic - Glass	4.30, 4.50	0.0035	10^{7}	10^{7}	17	15	0.06	0.43	9	9	40	>8
RF-60A	PTFE Ceramic - Glass	6.15[d]	0.0038[a]	10^{8}	10^{8}	>18	>15	0.02	0.40	9	8	69	8

$$\Delta \underline{\vec{H}} = -\omega^2 \mu \varepsilon \left(1 - j\frac{\sigma}{\varepsilon\omega}\right) \underline{\vec{H}}$$

$$= (-\omega^2 \mu \varepsilon)\left(-j\frac{\sigma}{\varepsilon\omega}\right)\left(1 - \frac{1}{j\frac{\sigma}{\varepsilon\omega}}\right)\underline{\vec{H}}$$

$$= j\omega\sigma\mu\underline{\vec{H}}$$

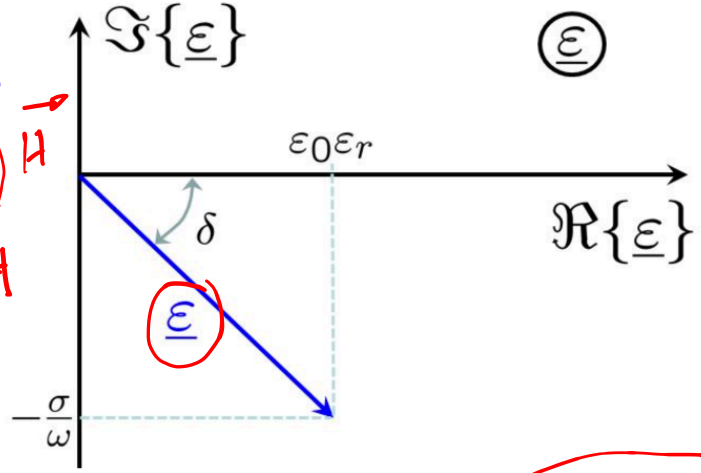


Figure 4.1: Complex permittivity $\underline{\varepsilon}$ and loss tangent $\tan\delta = \frac{\sigma}{\omega\varepsilon}$

$$\Delta \underline{\vec{H}} = -\omega^2 \mu \underline{\varepsilon}\, \underline{\vec{H}}$$

$$\Delta \underline{\vec{E}} = -\omega^2 \mu \underline{\varepsilon}\, \underline{\vec{E}}$$

$$\underline{\varepsilon} = \varepsilon\left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right) = \varepsilon\left(1 - j\frac{1}{\tau_{\text{relax}}\,\omega}\right)$$

$$\left[\varepsilon - j\left(\frac{\sigma}{\omega}\right)\right]$$

$(\varepsilon_0\varepsilon_r)$

For a **good conducting material** (i.e., metals, for example, silver, copper, gold, aluminium, brass, ..) we can neglect the real part of $\underline{\varepsilon}$ up to highest millimeter wave frequencies (because $\sigma \gg \omega\varepsilon$) and obtain

$$\Delta \underline{\vec{H}} = j\omega\mu\sigma\underline{\vec{H}} \qquad (4.25)$$

$$\Delta \underline{\vec{E}} = j\omega\mu\sigma\underline{\vec{E}} \qquad (4.26)$$

The same notation also holds for the current density ($\underline{\vec{J}} = \sigma\underline{\vec{E}}$):

$$\Delta \underline{\vec{J}} = j\omega\mu\sigma\underline{\vec{J}} \qquad (4.27)$$

Solutions of Wave Equation

$$\boxed{\Delta \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$

with:

$$\Delta \vec{E} = \begin{pmatrix} \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \\ \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \\ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \end{pmatrix}$$

Electromagnetic Waves

$$\begin{pmatrix} \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \\ \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \\ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \end{pmatrix} = \mu\sigma \begin{pmatrix} \frac{\partial E_x}{\partial t} \\ \frac{\partial E_y}{\partial t} \\ \frac{\partial E_z}{\partial t} \end{pmatrix} + \mu\varepsilon \begin{pmatrix} \frac{\partial^2 E_x}{\partial t^2} \\ \frac{\partial^2 E_y}{\partial t^2} \\ \frac{\partial^2 E_z}{\partial t^2} \end{pmatrix}$$

$\Delta \vec{E}$ $\qquad \frac{\partial}{\partial t}\vec{E}$ $\qquad \frac{\partial^2}{\partial t^2}\vec{E}$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

$\boxed{(\text{for } \sigma = 0)}$ $\boxed{\text{insulator}}$

Electromagnetic Waves

For solving this PDE

d'Alembert:

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

function

cos

$$\boxed{E_x = f(\omega t - \vec{k} \cdot \vec{r}) + g(\omega t + \vec{k} \cdot \vec{r})}$$

$x^2 + 5x + 6 = 0$

$x = 0, 0$

Wave Vector:

$$\vec{k} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$$

Wave Number:

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$\boxed{k = \omega\sqrt{\mu\varepsilon}} = \frac{\omega}{c_0}\sqrt{\mu_r \varepsilon_r}$$

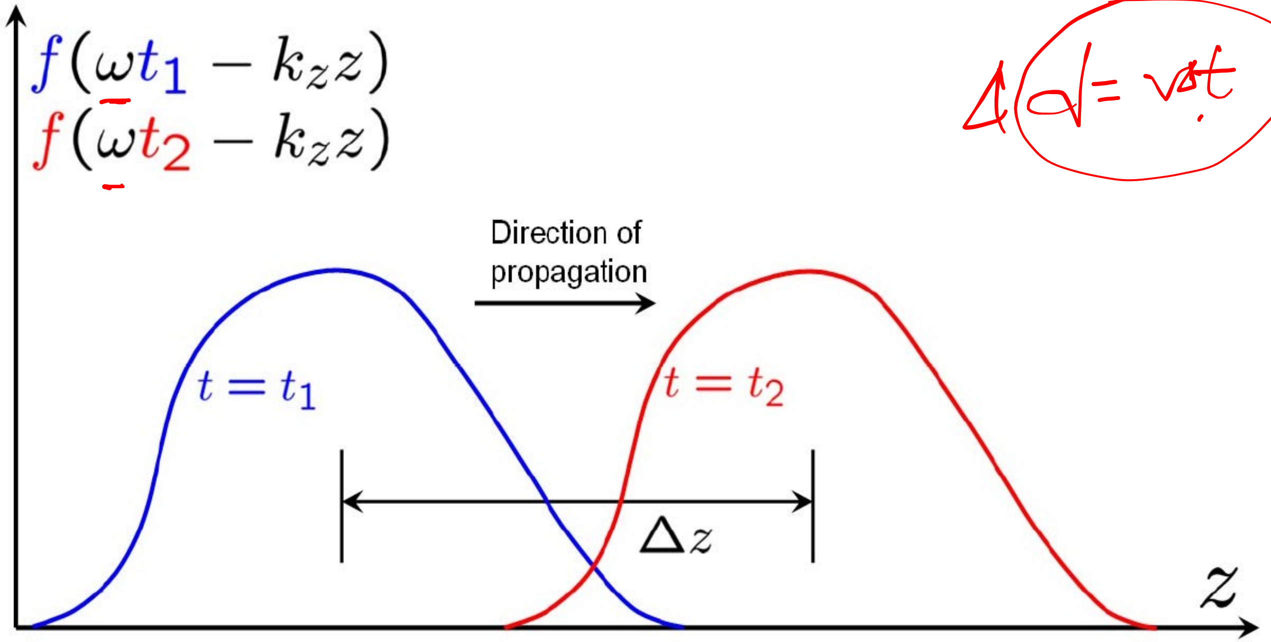
- In physics, a wave vector is a vector which helps describe a wave. Like any vector, it has a magnitude and direction, both of which are important.
- Its magnitude is either the wavenumber or angular wavenumber of the wave, and
- its direction is ordinarily the direction of wave propagation.

Electromagnetic Waves

c_0

$0.9c_0$

Wave propagation:



$\Delta d = v\Delta t$

$$\boxed{E_x = f(\omega t - \vec{k} \cdot \vec{r})}$$

$$\vec{k} \cdot \vec{r} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_x x + k_y y + k_z z$$

$$E_x = f(\omega t - k_x x - k_y y - k_z z)$$

If this wave travels into the positive z-direction, we get

w.v

w.n

$$E_x = f(\omega t - k_z z) \quad \text{with} \quad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ k_z \end{pmatrix} \quad \text{and} \quad |\vec{k}| = k = k_z$$

$$\Delta z = (t_2 - t_1)\frac{c_0}{\sqrt{\mu_r \varepsilon_r}}$$

$$= (t_2 - t_1) v_{ph}$$

$$v_{ph} = \frac{c_0}{\sqrt{\mu_r \varepsilon_r}}$$

Therefore, in a medium, electromagnetic waves propagate slower than in vacuum.

Electromagnetic (lossy and lossless) Waves

Wave Equation in Complex Notation:

$$\Delta \underline{\vec{H}} = -\omega^2 \mu \underline{\varepsilon}\, \underline{\vec{H}}$$

$$\Delta \underline{\vec{E}} = -\omega^2 \mu \underline{\varepsilon}\, \underline{\vec{E}}$$

$$\Delta \vec{H} = \begin{pmatrix} \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} \\ \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} \\ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} \end{pmatrix}$$

with

$$\begin{aligned} \underline{\varepsilon} &= \varepsilon \left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right) \\ &= \varepsilon \left(1 - j\frac{1}{\tau_{\mathrm{relax}}\,\omega}\right) \\ &= \varepsilon \cdot (1 - j\ \tan\delta) \end{aligned}$$

$$\tan\delta = \frac{\sigma}{\omega\,\varepsilon} \qquad \text{with} \quad \varepsilon = \varepsilon_0\,\varepsilon_r$$

Loss Tangent
(Dissipation Factor)

Electromagnetic (lossy and lossless) Waves

$\Delta \underline{\vec{E}} = -\omega^2 \mu \underline{\varepsilon} \underline{\vec{E}}$

$$\Delta \underline{\vec{E}} + \omega^2 \mu \underline{\varepsilon}\, \underline{\vec{E}} = 0 \qquad \textbf{Wave Equation}$$

$$\vec{E}(x,y,z,t) = \Re\{\underline{\vec{E}}_0(x,y,z)\, \mathrm{e}^{\,j(\omega t - \underline{\vec{k}}\cdot\vec{r})}\}$$

$t = 0$

$$\underline{\vec{E}}(x,y,z) = \underline{\vec{E}}_0(x,y,z)\, \mathrm{e}^{\,-j\underline{\vec{k}}\cdot\vec{r}}$$

[Without considering time dependency]

complex wave vector

$$\underline{\vec{k}} = \begin{pmatrix} \underline{k}_x \\ \underline{k}_y \\ \underline{k}_z \end{pmatrix}$$

complex wave number

$$\begin{aligned} \underline{k} &= |\underline{\vec{k}}| \\ &= \omega\sqrt{\mu\underline{\varepsilon}} \\ &= \omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right)} \\ &= \omega\sqrt{\mu\varepsilon}\sqrt{(1 - j\tan\delta)} \end{aligned}$$

Electromagnetic Waves

$$\underline{k} = \omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right)}$$

Insulator: $\sigma = 0$

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$k_{\text{insulator}} = \omega\sqrt{\mu\,\varepsilon}$$

$$= \frac{\omega}{c_0}\sqrt{\mu_r\,\varepsilon_r}$$

$$= \frac{\omega}{v_{ph}}$$

Electromagnetic Waves

Good electric conductor: $\sigma \gg \varepsilon\omega$

$$\underline{k} = \omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right)}$$

$= \omega\sqrt{-j\mu\epsilon\frac{\sigma}{\epsilon\omega}\left(1 - \frac{1}{j\frac{\sigma}{\epsilon\omega}}\right)}$ 0

$= \sqrt{-j\mu\epsilon \times \frac{\sigma}{\epsilon\omega} \times \omega^2}$

But for a **good electric conductor** ($\sigma \gg \varepsilon\omega$) the wave number becomes

$$\underline{k}_{\text{ conductor}} = \sqrt{-j\omega\mu\sigma}$$

and with $\boxed{\sqrt{-j} = (1-j)/\sqrt{2}}$ we get

$$\underline{k}_{\text{ conductor}} = (1-j)\sqrt{\frac{\omega\mu\sigma}{2}} \quad (= k' - jk'')$$

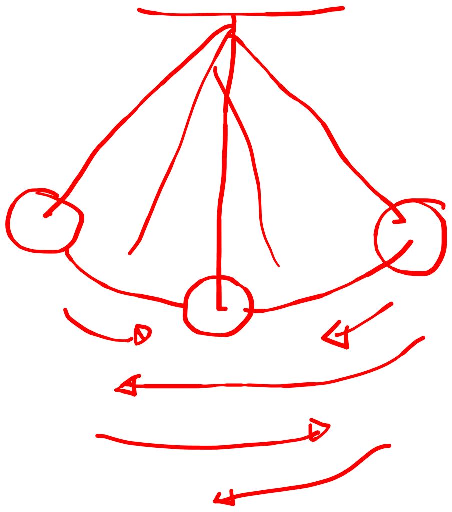
with the phase constant k'

$$k' = \sqrt{\frac{\omega\mu\sigma}{2}}$$

and the so-called damping constant k'' (here being equal to k')

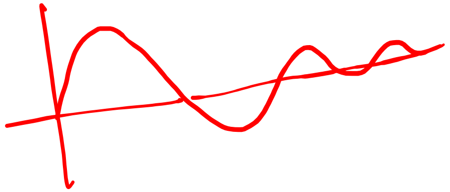
$$k'' = \sqrt{\frac{\omega\mu\sigma}{2}}$$

Good electric conductor: $\sigma \gg \varepsilon\omega$



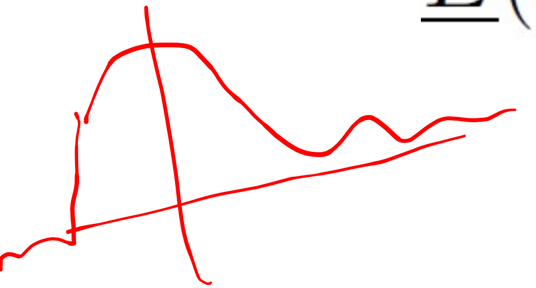
with the phase constant k'

$$k' = \sqrt{\frac{\omega\mu\sigma}{2}}$$



and the so-called damping constant k'' (here being equal to k')

$$k'' = \sqrt{\frac{\omega\mu\sigma}{2}}$$



$$\begin{aligned}\underline{\vec{E}}(x,y,z) &= \underline{\vec{E}}_0(x,y,z)\,\mathrm{e}^{-j(1-j)\sqrt{\frac{\omega\mu\sigma}{2}}\vec{n}\cdot\vec{r}} \\ &= \underline{\vec{E}}_0(x,y,z)\,\underbrace{\mathrm{e}^{-\frac{1}{\delta_s}\vec{n}\cdot\vec{r}}}_{\text{damping factor}}\;\underbrace{\mathrm{e}^{-j\frac{1}{\delta_s}\vec{n}\cdot\vec{r}}}_{\text{phase factor}}\end{aligned}$$

Poor electric conductor:

$$\underline{k}_{\text{poor cond.}} = \omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right)}$$

$$= \omega\sqrt{\mu\varepsilon}\,\sqrt{1 - j\frac{\sigma}{\varepsilon\,\omega}}$$

$$= \omega\sqrt{\mu\varepsilon}\,\sqrt{1 - j\tan\delta}$$

$$\sqrt{1 - jx} \approx 1 - j\frac{x}{2} \quad \text{for } |x| \ll 1$$

$(1-x)^{-2} \approx 1 - x(-2) \approx 1 + 2x$

$(1-jx)^{1/2} \approx 1 - jx \times \frac{1}{2} \approx 1 - j\frac{x}{2}$

$\underline{k}_{poor} = \underbrace{\omega\sqrt{\mu\varepsilon}}_{k'}\left[1 - j\frac{\sigma}{2\varepsilon\omega}\right]$ k''

we get

$$\underline{k}_{\text{poor cond.}} \approx \omega\sqrt{\mu\varepsilon} - j\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \quad (= k' - jk'')$$

with the phase constant k'

$$k' = \omega\sqrt{\mu\varepsilon}$$

and the damping constant k''

$$k'' = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

Poor electric conductor:

we get

$$\underline{k}_{\text{ poor cond.}} \approx \omega\sqrt{\mu\varepsilon} - j\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \quad (= k' - jk'')$$

with the phase constant k'

$$k' = \omega\sqrt{\mu\varepsilon}$$

and the damping constant k''

$$k'' = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$\begin{aligned}
\underline{\vec{E}}(x,y,z) &= \underline{\vec{E}}_0(x,y,z)\, \mathrm{e}^{-j\left(\omega\sqrt{\mu\varepsilon} - j\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\right)\vec{n}\cdot\vec{r}} \\
&= \underline{\vec{E}}_0(x,y,z)\, \underbrace{\mathrm{e}^{-\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\,\vec{n}\cdot\vec{r}}}_{\text{damping factor}}\, \underbrace{\mathrm{e}^{-j\omega\sqrt{\mu\varepsilon}\,\vec{n}\cdot\vec{r}}}_{\text{phase factor}}
\end{aligned}$$

Skin Depths: Skin effect is the tendency of an alternating electric current to become distributed within a conductor such that the current density is largest near the surface of the conductor and decreases exponentially with greater depths in the conductor.

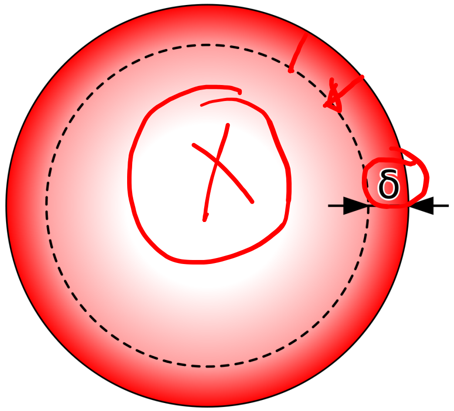


Figure: Skin Depth

$$\boxed{\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}}}$$

Good electric conductor (e.g., copper)

$$\delta_s(\text{copper}) \approx \frac{66.1\ \mu\text{m}}{\sqrt{f\,[\text{MHz}]}}$$

$$\delta_s = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

Poor electric conductor

Example: The skin depth of copper with $\sigma \approx 56 \cdot 10^6\ \text{S/m}$ is

$$\delta_s(\text{copper}) \approx \frac{67\ \mu\text{m}}{\sqrt{f\,[\text{MHz}]}}$$

This yields a skin depth of $67\ \mu\text{m}$ at $1\ \text{MHz}$, $6.7\ \mu\text{m}$ at $100\ \text{MHz}$ and $0.67\ \mu\text{m}$ at $10\ \text{GHz}$. At $50\ \text{Hz}$ we get a skin depth of $9.5\ \text{mm}$.