# Lecture 4

## Boundary Conditions

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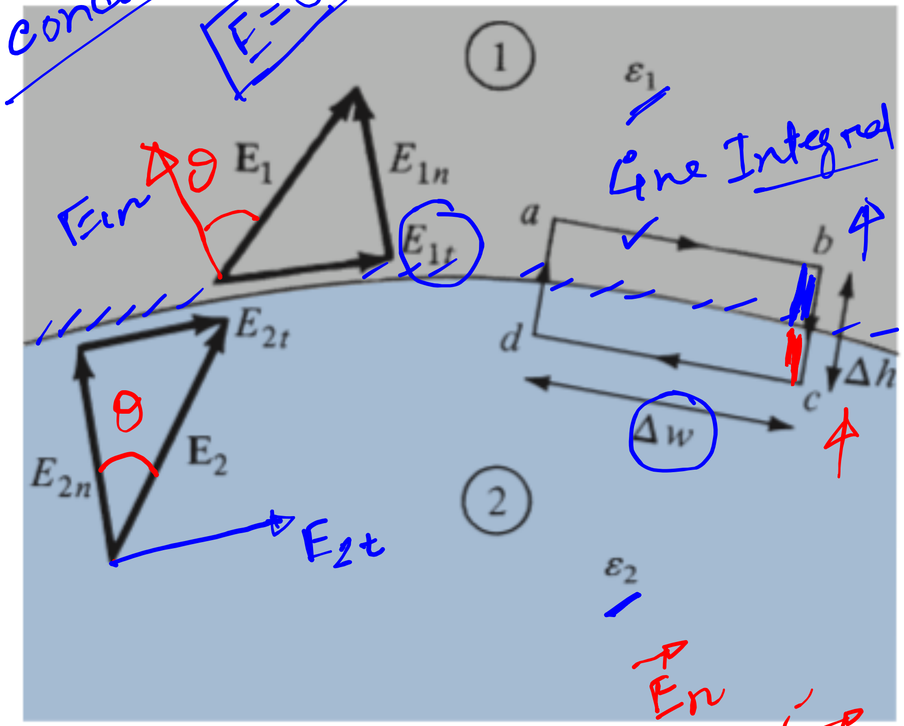
$E_{1n} \neq E_{2n}$

# Electric Boundary Conditions(Dielectric-Dielectric)

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$$

conductor

$\boxed{E=0}$




(a)

$$0 = E_{1t}\,\Delta w - \cancel{E_{1n}\frac{\Delta h}{2}} - \cancel{E_{2n}\frac{\Delta h}{2}} - E_{2t}\Delta w + \cancel{E_{2n}\frac{\Delta h}{2}} + \cancel{E_{1n}\frac{\Delta h}{2}}$$

where $E_t = |\mathbf{E}_t|$ and $E_n = |\mathbf{E}_n|$. The $\dfrac{\Delta h}{2}$ terms cancel, and eq. (5.56) becomes

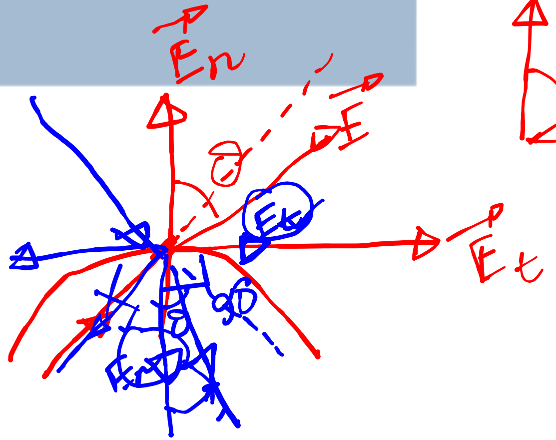
$$0 = (E_{1t} - E_{2t})\Delta w$$

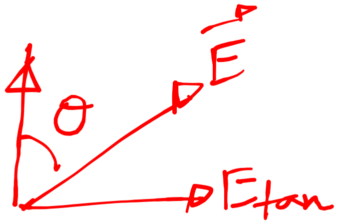
or

$\vec{E} = \vec{E}_{tan} + \vec{E}_n$

$D = \epsilon E$

$$\boxed{E_{1t} = E_{2t}}$$




$\theta$ = Normal ^ Vector

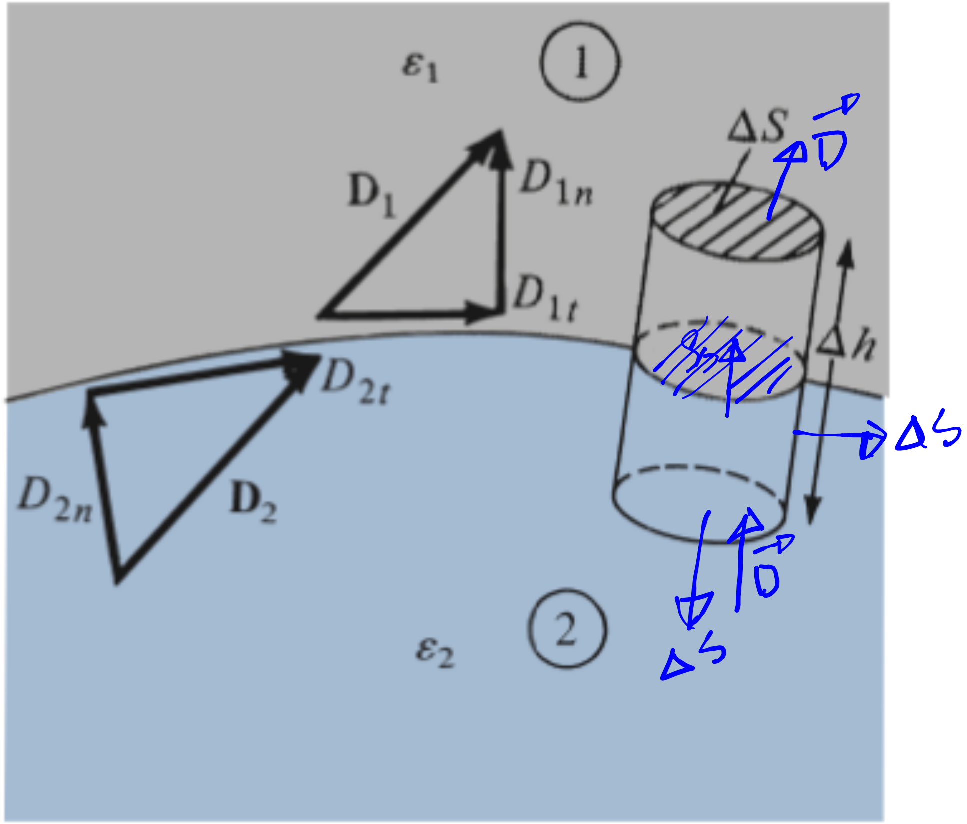
$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

# Electric Boundary Conditions (Dielectric-Dielectric)

$$\rho_s = \frac{Q_{enc}}{\Delta S}$$




(b)

$$\Delta Q = \rho_S \Delta S = D_{1n} \Delta S - D_{2n} \Delta S$$

$$Q = \rho_S \Delta S$$

$$\boxed{\cos 90 = 0}$$

$$\boxed{D_{1n} - D_{2n} = \rho_S}$$

where $\rho_S$ is the free charge density placed deliberately at the boundary. It should be borne in mind that eq. (5.59) is based on the assumption that **D** is directed from region 2 to region 1 and eq. (5.59) must be applied accordingly. If no free charges exist at the interface (i.e., charges are not deliberately placed there), $\rho_S = 0$ and eq. (5.59) becomes

$$\boxed{D_{1n} = D_{2n}} \tag{5.60}$$

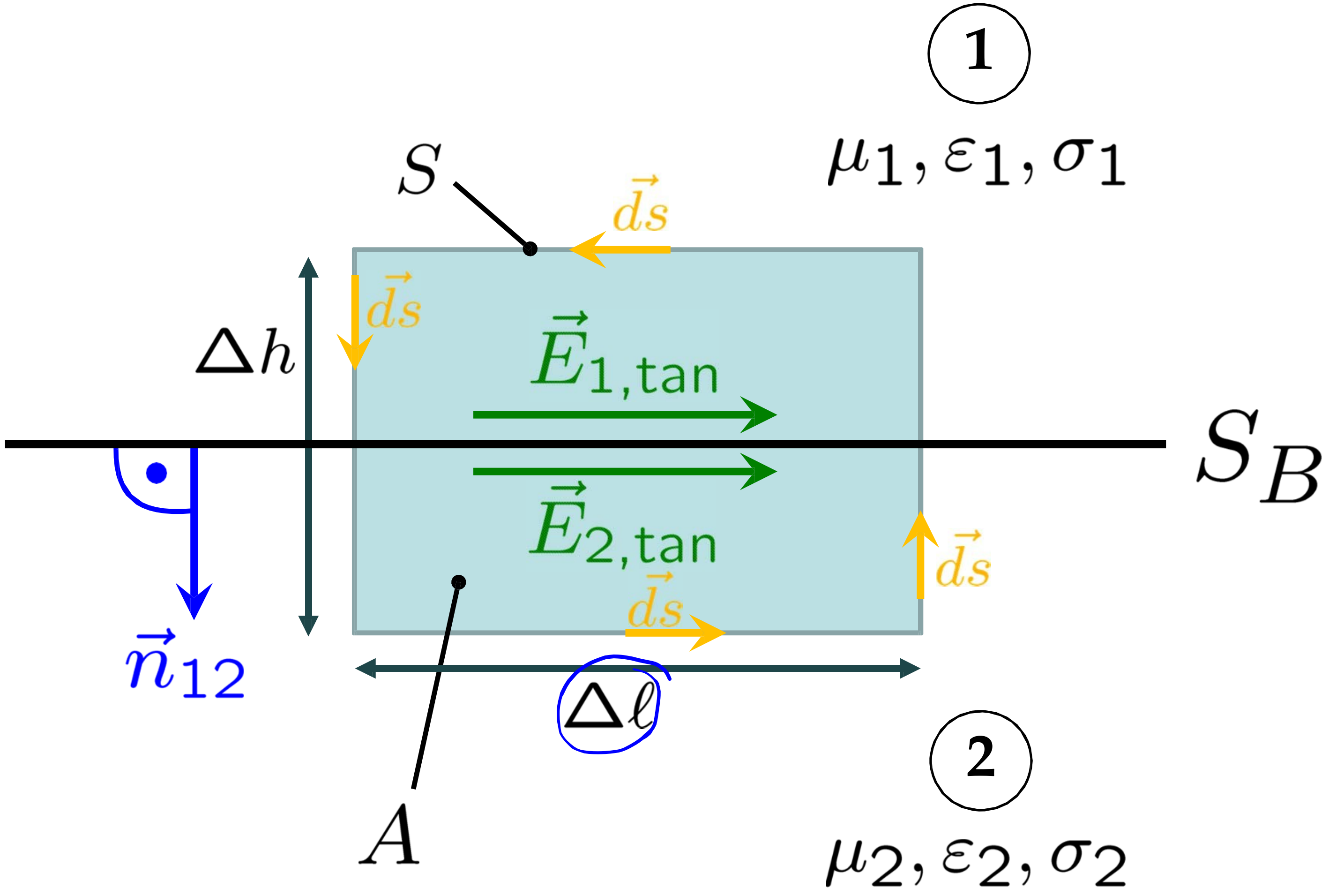
Thus the normal component of **D** is continuous across the interface; that is, $D_n$ undergoes no change at the boundary. Since $\mathbf{D} = \varepsilon \mathbf{E}$, eq. (5.60) can be written as

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \tag{5.61}$$

$$\Rightarrow E_{1n} = \varepsilon_2/\varepsilon_1 E_{2n}$$

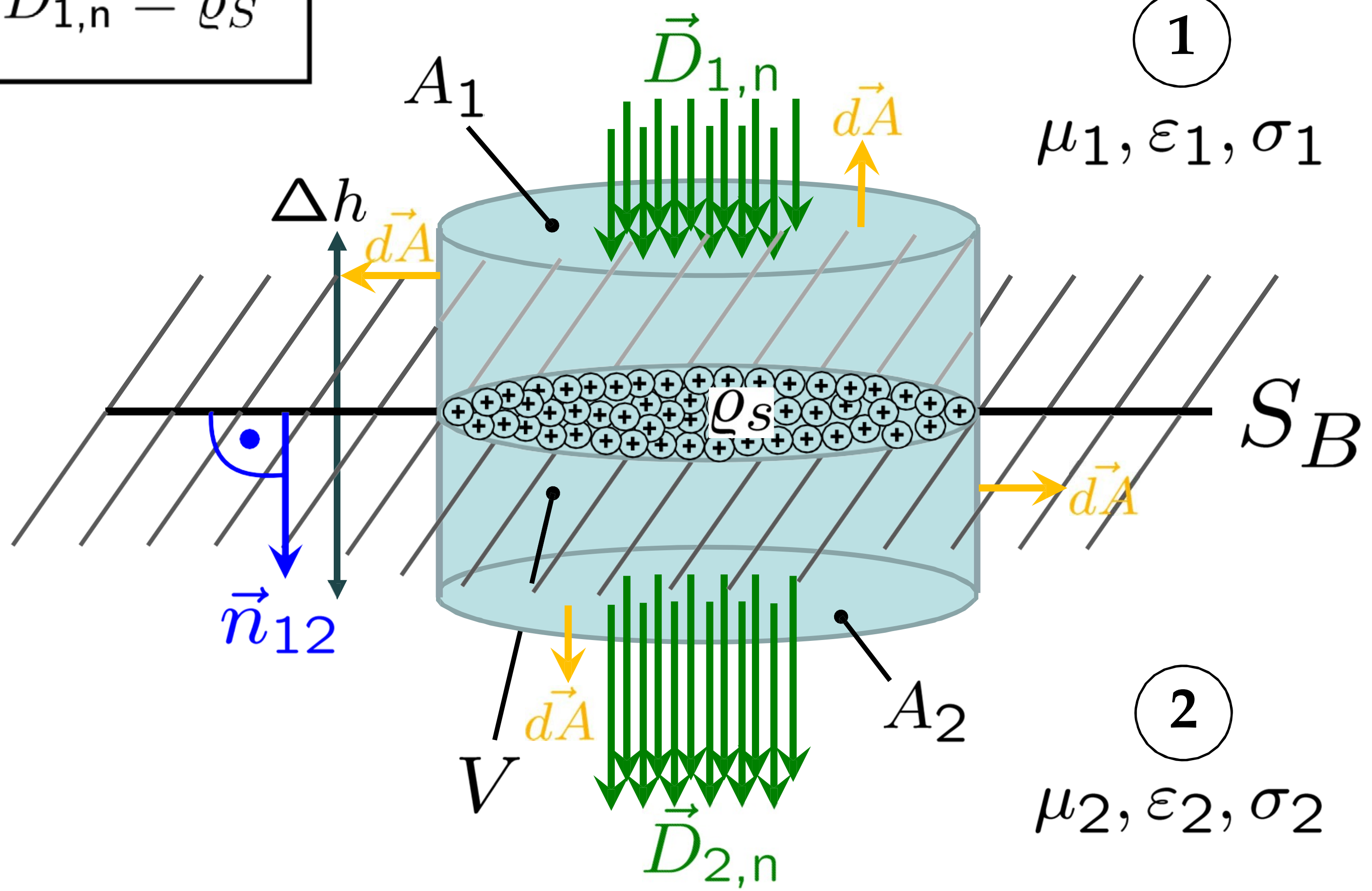
# Boundary Conditions

$$\boxed{\vec{E}_{2,\mathrm{tan}} = \vec{E}_{1,\mathrm{tan}}}$$



# Boundary Conditions

$$\boxed{D_{2,\mathrm{n}} - D_{1,\mathrm{n}} = \varrho_S}$$

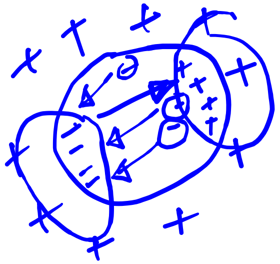


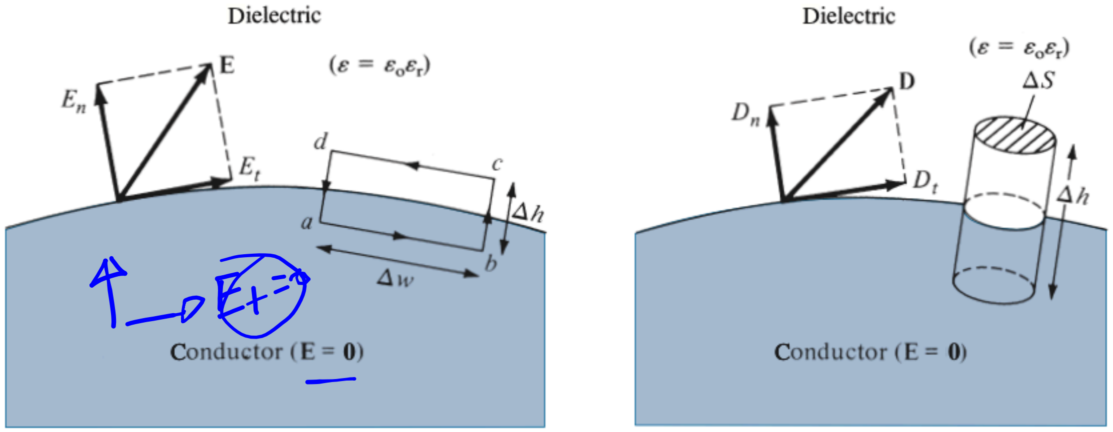
+1C

# Electric Boundary Conditions(Conductor-Dielectric)

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$






(a) (b)

abcda

$$0 = \cancel{0 \cdot \Delta w} + \cancel{0 \cdot \frac{\Delta h}{2}} + \cancel{E_n \cdot \frac{\Delta h}{2}} - E_t \cdot \Delta w - \cancel{E_n \cdot \frac{\Delta h}{2}} - \cancel{0 \cdot \frac{\Delta h}{2}}$$

As $\Delta h \to 0$,

$$E_t = 0$$

$$\boxed{E_{t1} = E_{t2} = 0}$$

$$D_2 \Delta S - \cancel{D_1 \Delta S} = 0$$

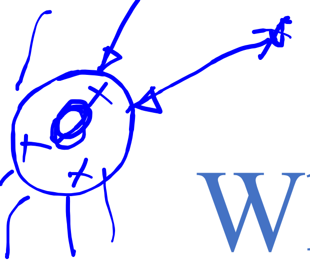
$$\Delta Q = D_n \cdot \Delta S - 0 \cdot \Delta S$$

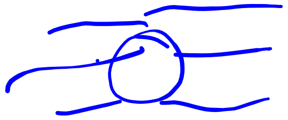
because $\mathbf{D} = \varepsilon\mathbf{E} = \mathbf{0}$ inside the conductor. Equation (5.68) may be written as

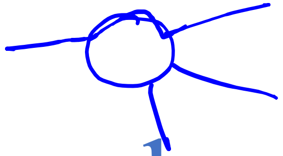
$$D_n = \frac{\Delta Q}{\Delta S} = \rho_S$$

or

$$D_n = \rho_S$$







# What is a perfect conductor?

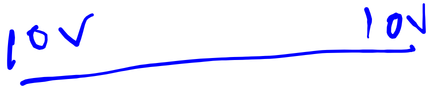
$R \propto L$

$R \propto \frac{1}{A}$

$R = \rho \frac{L}{A}$

Thus under static conditions, the following conclusions can be made about a perfect conductor:

1. No electric field may exist *within* a conductor; that is, considering our conclusion in Section 5.4,



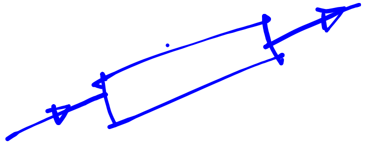

$\rho \frac{L}{A} = R$

(resistivity)

$E = -\nabla V$

$$\boxed{\rho_v = 0, \quad \mathbf{E} = \mathbf{0}} \tag{5.70}$$

2. Since $\mathbf{E} = -\nabla V = \mathbf{0}$, there can be no potential difference between any two points in the conductor; that is, a conductor is an equipotential body.
3. An electric field $\mathbf{E}$ must be external to the conductor and must be *normal* to its surface; that is,



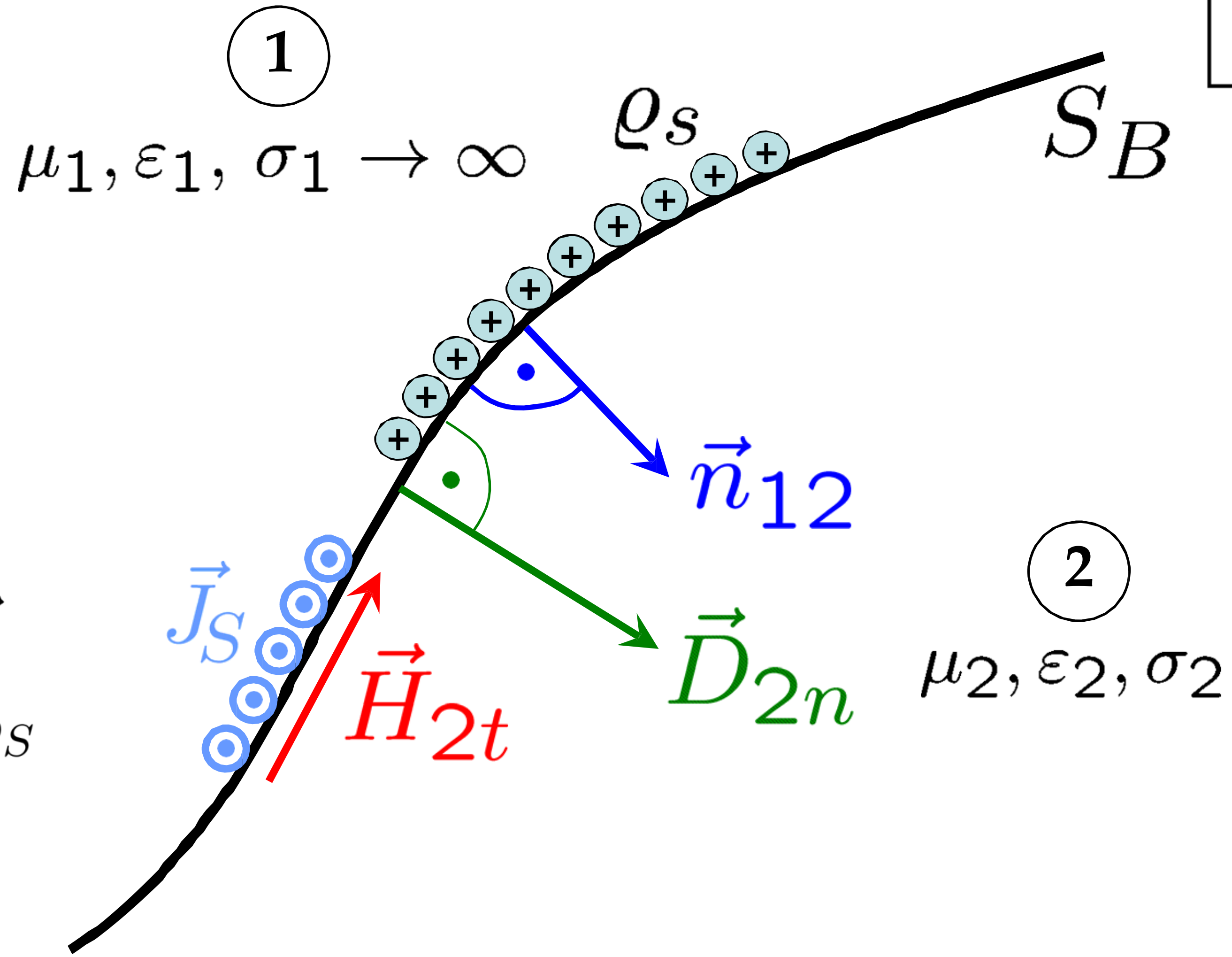
$$\boxed{D_t = \varepsilon_o \varepsilon_r E_t = 0, \quad D_n = \varepsilon_o \varepsilon_r E_n = \rho_S} \tag{5.71}$$

## Maxwell's Equations

$$\vec{E}_{2,\text{tan}} = \vec{E}_{1,\text{tan}}$$

**Boundary conditions at a perfect electric conductor:**

$$D_{2,\text{n}} - D_{1,\text{n}} = \varrho_S$$




$$\begin{aligned} \vec{E}_{2t} &= \vec{0} \\ D_{2n} &= \varrho_S \end{aligned}$$

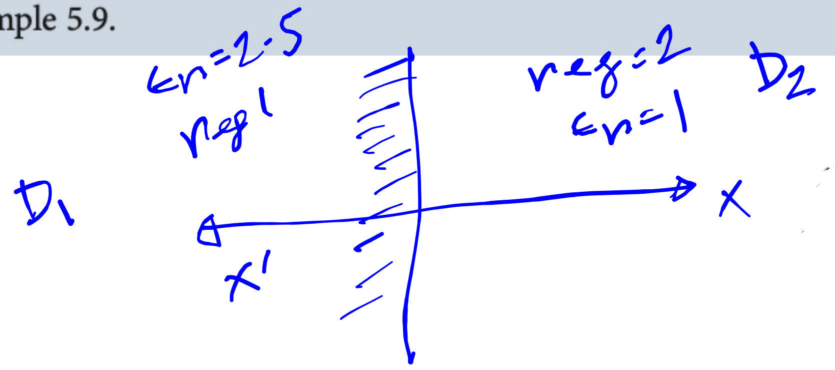
**PRACTICE EXERCISE 5.9**

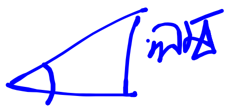
$\varepsilon_r = 1$

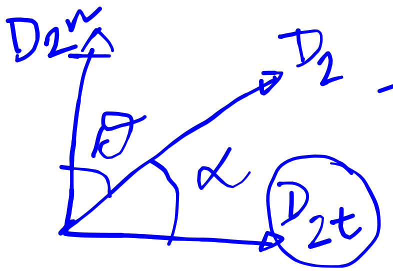
A homogeneous dielectric ($\varepsilon_r = 2.5$) fills region 1 ($x < 0$) while region 2 ($x > 0$) is free space.

(a) If $\mathbf{D}_1 = 12\mathbf{a}_x - 10\mathbf{a}_y + 4\mathbf{a}_z$ nC/m², find $\mathbf{D}_2$ and $\theta_2$.

(b) If $E_2 = 12$ V/m and $\theta_2 = 60°$, find $E_1$ and $\theta_1$. Take $\theta_1$ and $\theta_2$ as defined in Example 5.9.






$\tan\theta = \dfrac{D_{2t}}{D_{2n}}$

$$\tan\theta_2 = \frac{D_{2t}}{D_{2n}} = \frac{\sqrt{(-4)^2 + (1.6)^2}}{12} = 0.359 \longrightarrow \underline{\underline{\theta_2 = 19.75°}}$$

**P. E. 5.9 (a)** Since $a_n = a_x$,

$D_{2t}$

tangential angle, $\alpha = 90° - \theta$

$$\vec{D}_{1n} = 12\hat{a}_x, \qquad D_{1t} = -10a_x + 4a_z, \qquad D_{2n} = D_{1n} = 12a_x$$

$$E_{2t} = E_{1t} \longrightarrow D_{2t} = \frac{\varepsilon_2 D_{1t}}{\varepsilon_1} = \frac{1}{2.5}(-10a_y + 4a_z) = -4a_y + 1.6a_z$$

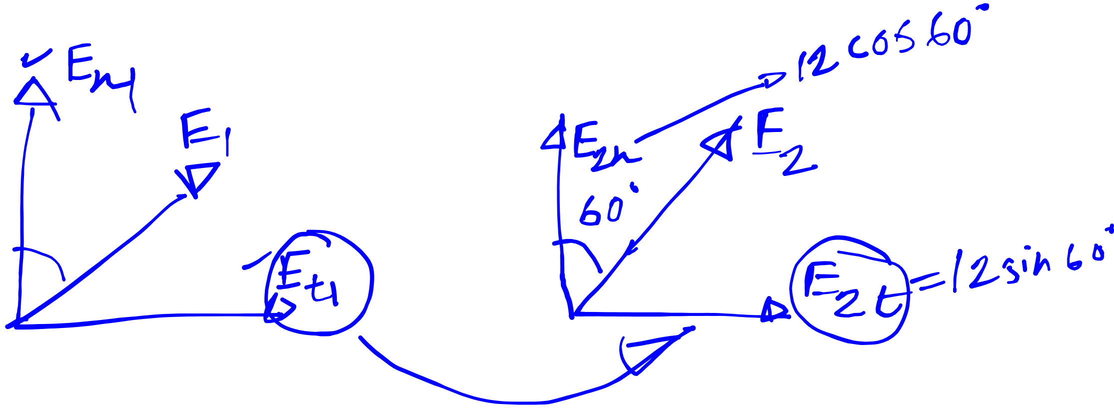
$$D_2 = D_{2n} + D_{2t} = \underline{\underline{-12a_x - 4a_y + 1.6a_z}} \text{ nC/m}^2.$$

**PRACTICE EXERCISE 5.9**

A homogeneous dielectric ($\varepsilon_r = 2.5$) fills region 1 ($x < 0$) while region 2 ($x > 0$) is free space.

(a) If $\mathbf{D}_1 = 12\mathbf{a}_x - 10\mathbf{a}_y + 4\mathbf{a}_z$ nC/m$^2$, find $\mathbf{D}_2$ and $\theta_2$.

(b) If $E_2 = 12$ V/m and $\theta_2 = 60°$, find $E_1$ and $\theta_1$. Take $\theta_1$ and $\theta_2$ as defined in Example 5.9.



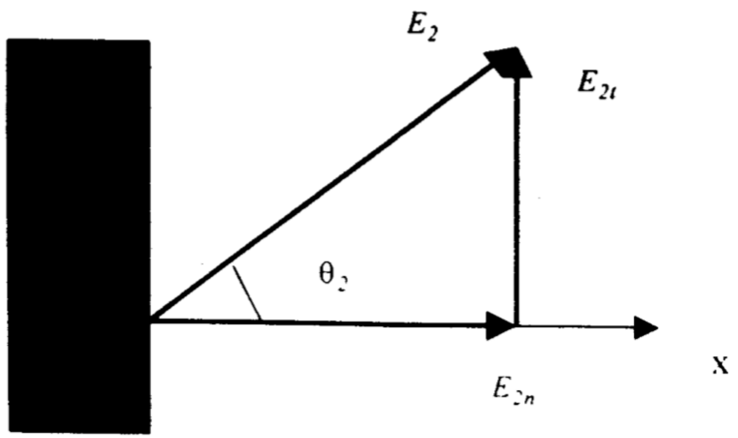

$E_2 = - -$

$D_{1n} = D_{2n}$

$\epsilon E_{1n} = \epsilon$

$E_1$

(b) $E_{1t} = E_{2t} = E_2 \sin\theta_2 = 12\sin 60^o = 10.392$




$$E_{1n} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} E_{2n} = \frac{1}{2.5} 12\cos 60^o = 2.4$$

$$E_1 = \sqrt{E_{1t}^2 + E_{1n}^2} = \underline{\underline{10.67}}$$

$$\tan\theta_1 = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \tan\theta_2 = \frac{2.5}{1} \tan 60^o = 4.33 \longrightarrow \underline{\underline{\theta_1 = 77^o}}$$

Note that $\theta_1 > \theta_2$.

### 3.3 Problem 3

$D_{1n} = D_{2n}$

The three dimensional space is split into two regions filled with different materials (see Fig. 3.3.2). The electric vector field $\vec{E}_1$ and the magnetic vector field $\vec{H}_1$, in region 1 (see Fig. 3.3.2), are given as

$$\vec{E}_1 = (2.0\vec{e}_y + 3.0\vec{e}_z)\,\mathrm{V/m} \qquad \vec{H}_1 = (0.1\vec{e}_x + 0.2\vec{e}_z)\,\mathrm{A/m}$$

For sub-problems a) and b) there are no surface charges $\varrho_s$ on the interface.

a) Find the electric vector field $\vec{E}_2$ in region 2 so that the boundary conditions at the interface are fulfilled.

b) Find the magnetic induction $\vec{B}_2$ in region 2 so that the boundary conditions at the interface are fulfilled.

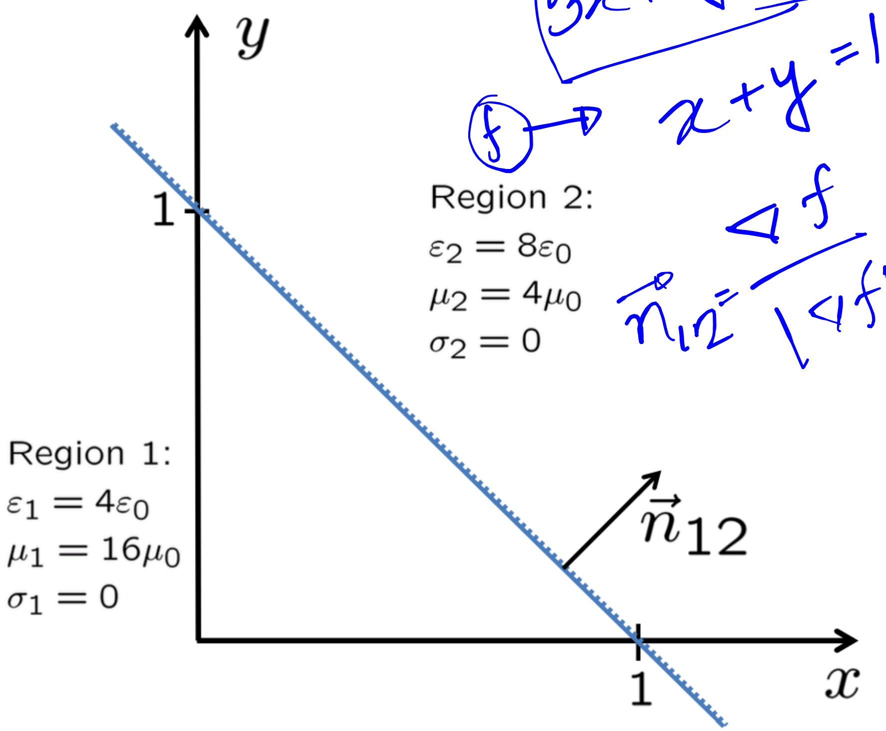
c) Now the medium in region 2 is replaced by an ideal conductor with conductivity of $\sigma_2 \to \infty$ while all other parameters remain the same. Determine the surface charge density $\varrho_S$ on the interface and determine the surface current density $\vec{J}_S$.

$\boxed{3x + 4y \leq 12}$

$f \rightarrow x + y = 1$

$\vec{n}_{12} = \dfrac{\nabla f}{|\nabla f|}$

$\dfrac{\hat{e}_x + \hat{e}_y}{\sqrt{2}}$




a) Normal vector: $\vec{n}_{12} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$

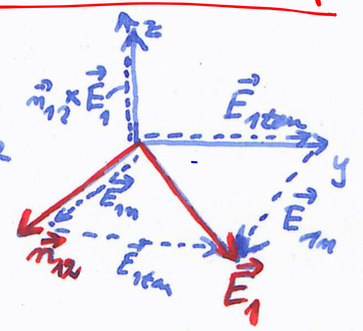
$\vec{E}_1 = \begin{pmatrix}0\\2\\3\end{pmatrix}\frac{\mathrm{V}}{\mathrm{m}}$ ; $\vec{H}_1 = \begin{pmatrix}0.1\\0\\0.2\end{pmatrix}\frac{\mathrm{A}}{\mathrm{m}}$

$\vec{E}_{2tan} = \vec{E}_{1tan}$

Please note: $\vec{n}_{12} \cdot \vec{E}_1 = |\vec{E}_{1n}|$ ; $\boxed{\vec{E}_{1n} = (\vec{n}_{12} \cdot \vec{E}_1)\,\vec{n}_{12}}$

But: $\vec{n}_{12} \times \vec{E}_1 \neq \vec{E}_{1tan}$

Instead: $\vec{E}_{1tan} = \vec{n}_{12} \times \vec{E}_1 \times \vec{n}_{12}$




Here: $\vec{n}_{12} \times \vec{E}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\2\\3\end{pmatrix}\frac{\mathrm{V}}{\mathrm{m}} = \frac{1}{\sqrt{2}}\begin{pmatrix}3\\-3\\2\end{pmatrix}\frac{\mathrm{V}}{\mathrm{m}}$

$\vec{E}_{1tan} = \vec{n}_{12} \times \vec{E}_1 \times \vec{n}_{12} = \frac{1}{\sqrt{2}}\begin{pmatrix}3\\-3\\2\end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}\frac{\mathrm{V}}{\mathrm{m}} = \frac{1}{2}\begin{pmatrix}-2\\2\\6\end{pmatrix}\frac{\mathrm{V}}{\mathrm{m}} = \begin{pmatrix}-1\\1\\3\end{pmatrix}\frac{\mathrm{V}}{\mathrm{m}}$

$\Rightarrow \vec{E}_{2tan} = \begin{pmatrix}-1\\1\\3\end{pmatrix}\frac{\mathrm{V}}{\mathrm{m}}$

10

$|\vec{E}_{1n}| = \vec{n}_{12} \cdot \vec{E}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}0\\2\\3\end{pmatrix}\frac{\mathrm{V}}{\mathrm{m}} = \frac{2}{\sqrt{2}}\frac{\mathrm{V}}{\mathrm{m}} = \sqrt{2}\,\frac{\mathrm{V}}{\mathrm{m}}$

$\vec{E}_{1n} = (\vec{n}_{12} \cdot \vec{E}_1)\,\vec{n}_{12} = \begin{pmatrix}1\\1\\0\end{pmatrix}\frac{\mathrm{V}}{\mathrm{m}}$

## 3.3 Problem 3

The three dimensional space is split into two regions filled with different materials (see Fig. 3.3.2). The electric vector field $\vec{E}_1$ and the magnetic vector field $\vec{H}_1$, in region 1 (see Fig. 3.3.2), are given as

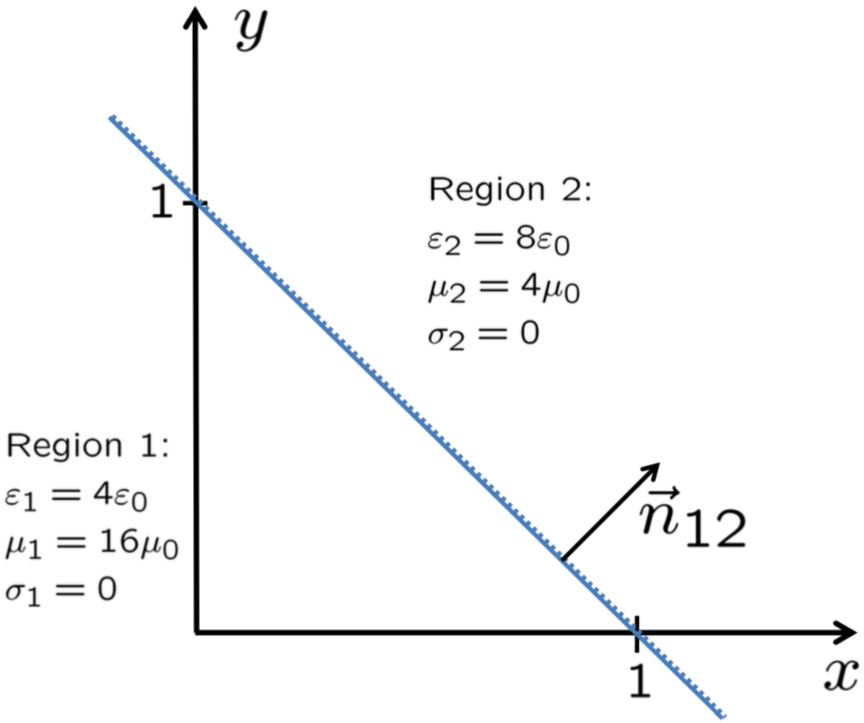
$$\vec{E}_1 = (2.0\vec{e}_y + 3.0\vec{e}_z)\,\mathrm{V/m} \qquad \vec{H}_1 = (0.1\vec{e}_x + 0.2\vec{e}_z)\,\mathrm{A/m}$$

For sub-problems a) and b) there are no surface charges $\varrho_s$ on the interface.

a) Find the electric vector field $\vec{E}_2$ in region 2 so that the boundary conditions at the interface are fulfilled.

b) Find the magnetic induction $\vec{B}_2$ in region 2 so that the boundary conditions at the interface are fulfilled.

c) Now the medium in region 2 is replaced by an ideal conductor with conductivity of $\sigma_2 \to \infty$ while all other parameters remain the same. Determine the surface charge density $\varrho_S$ on the interface and determine the surface current density $\vec{J}_S$.




$$\Rightarrow \vec{E}_{2tan} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \frac{V}{m}$$

10

$$|\vec{E}_{1n}| = \vec{n}_{12} \cdot \vec{E}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \frac{V}{m} = \frac{2}{\sqrt{2}} \frac{V}{m} = \sqrt{2} \frac{V}{m}$$

$$\vec{E}_{1n} = (\vec{n}_{12} \cdot \vec{E}_1)\vec{n}_{12} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \frac{V}{m}$$

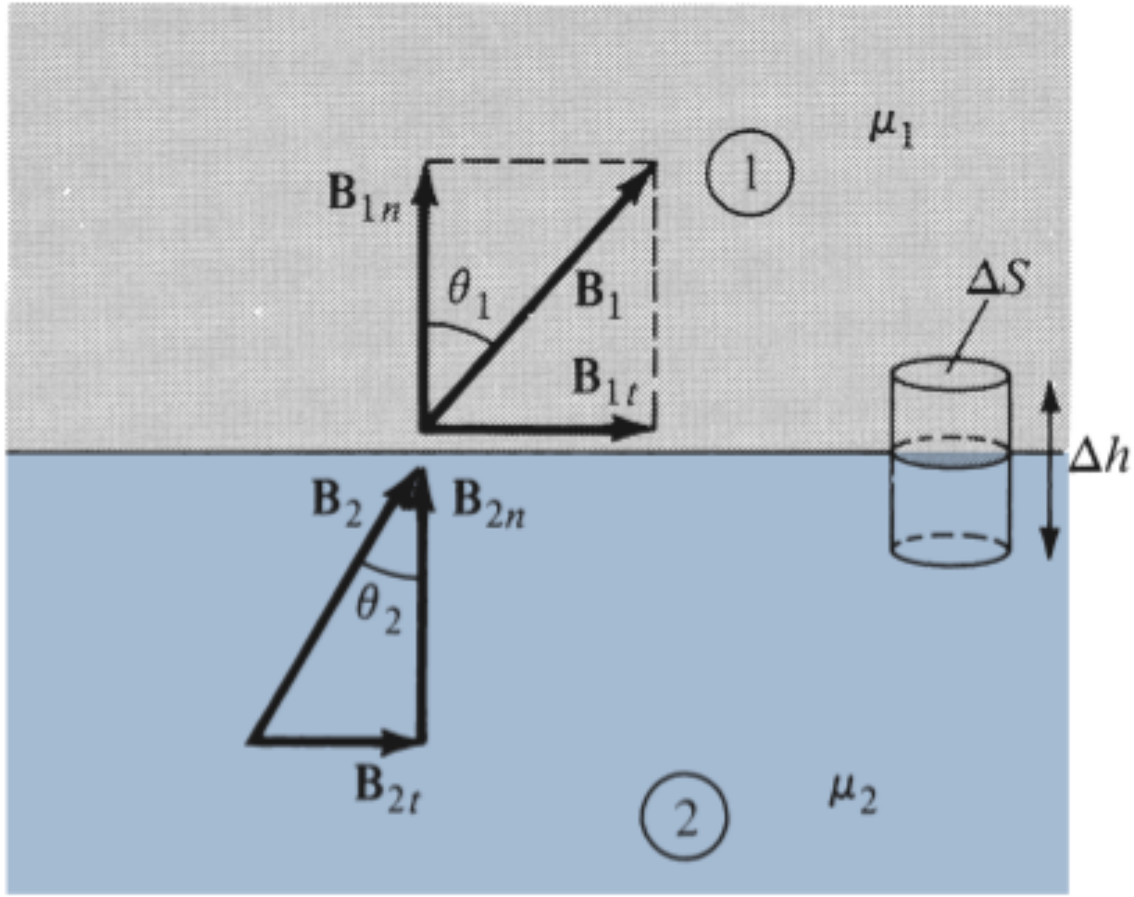
$$D_{2n} = D_{1n} \Rightarrow \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n} \Rightarrow E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{1}{2} E_{1n}$$

$$\Rightarrow \vec{E}_{2n} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \frac{V}{m}$$

$$\vec{E}_2 = \vec{E}_{2tan} + \vec{E}_{2n} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \frac{V}{m} + \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \frac{V}{m} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 3 \end{pmatrix} \frac{V}{m}$$

# Magnetic Boundary Conditions

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$




(a)

$$B_{1n}\,\Delta S - B_{2n}\,\Delta S = 0$$

$$\boxed{\mathbf{B}_{1n} = \mathbf{B}_{2n}} \quad \text{or} \quad \mu_1\mathbf{H}_{1n} = \mu_2\mathbf{H}_{2n}$$

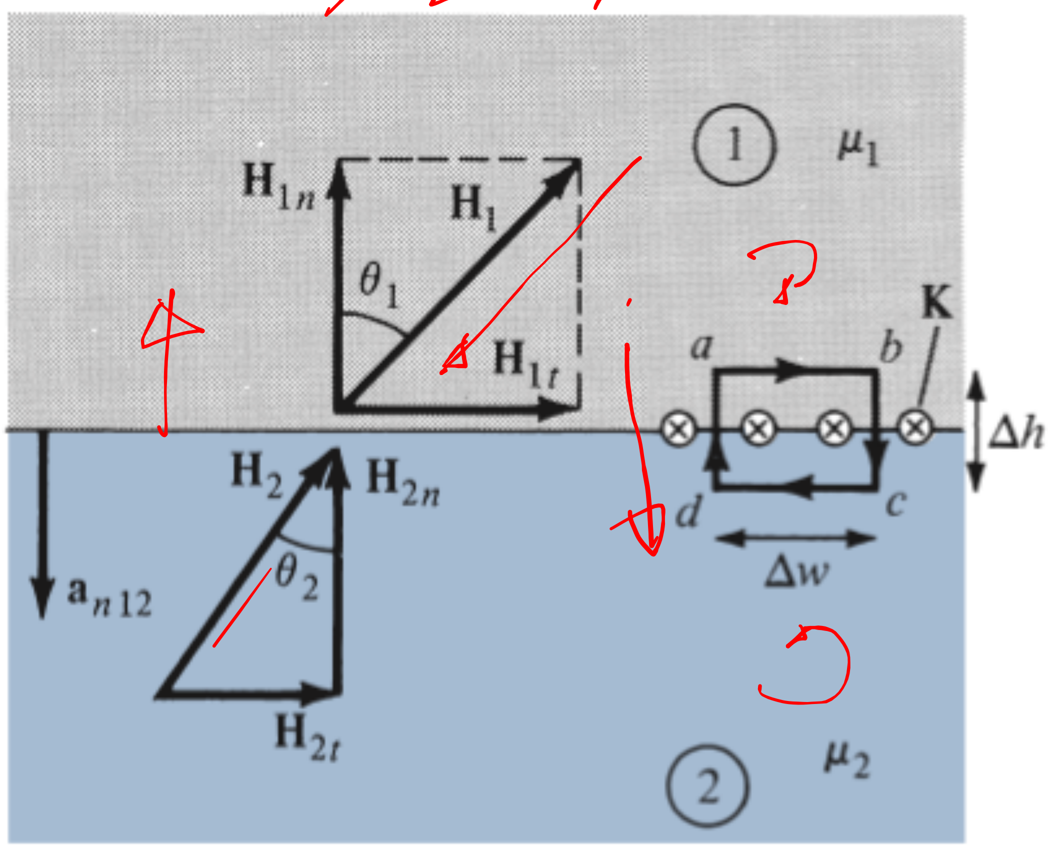
# Magnetic Boundary Conditions

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = I$$

$\boxed{\vec{H}_{1t} - \vec{H}_{2t} = \vec{K}}$

$H_{1t} \times \Delta W - H_{1n} \times \frac{\Delta h}{2} - H_{2n} \times \frac{\Delta h}{2} - H_{2t} \times \Delta W$
$+ H_{1n} \times \frac{\Delta h}{2} + H_{2n} \times \frac{\Delta h}{2}$

$+ - H_2$




(b)

Similarly, we apply eq. (8.39) to the closed path *abcda* of Figure 8.16(b), where surface current **K** on the boundary is assumed normal to the path. We obtain

A/m

$$K \cdot \Delta w = H_{1t} \cdot \Delta w + H_{1n} \cdot \frac{\Delta h}{2} + H_{2n} \cdot \frac{\Delta h}{2}$$

As $\Delta h \to 0$, eq. (8.42) leads to

$$H_{1t} - H_{2t} = K \tag{8.43}$$

This shows that the tangential component of $H$ is also discontinuous. Equation (8.43) may be written in terms of $B$ as

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K \tag{8.44}$$

In the general case, eq. (8.43) becomes

$$\boxed{(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}} \tag{8.45}$$

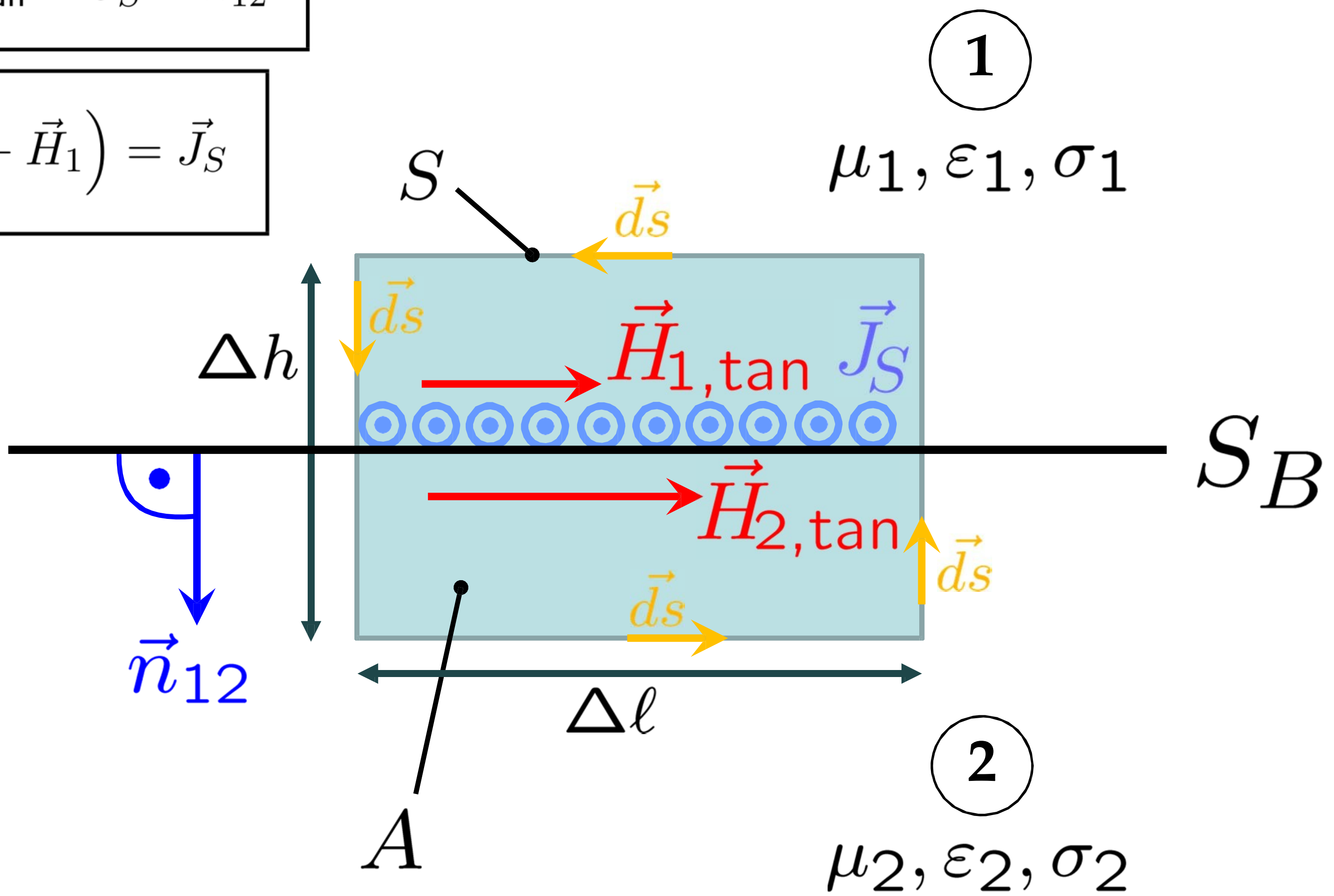
where $\mathbf{a}_{n12}$ is a unit vector normal to the interface and is directed from medium 1 to medium 2. If the boundary is free of current or the media are not conductors (for **K** is free current density), $\mathbf{K} = \mathbf{0}$ and eq. (8.43) becomes

$$\boxed{\mathbf{H}_{1t} = \mathbf{H}_{2t}} \qquad \text{or} \qquad \frac{\mathbf{B}_{1t}}{\mu_1} = \frac{\mathbf{B}_{2t}}{\mu_2} \tag{8.46}$$

# Boundary Conditions

$$\boxed{\vec{H}_{2,\text{tan}} - \vec{H}_{1,\text{tan}} = \vec{J}_S \times \vec{n}_{12}}$$

$$\boxed{\vec{n}_{12} \times \left(\vec{H}_2 - \vec{H}_1\right) = \vec{J}_S}$$



## Boundary Conditions

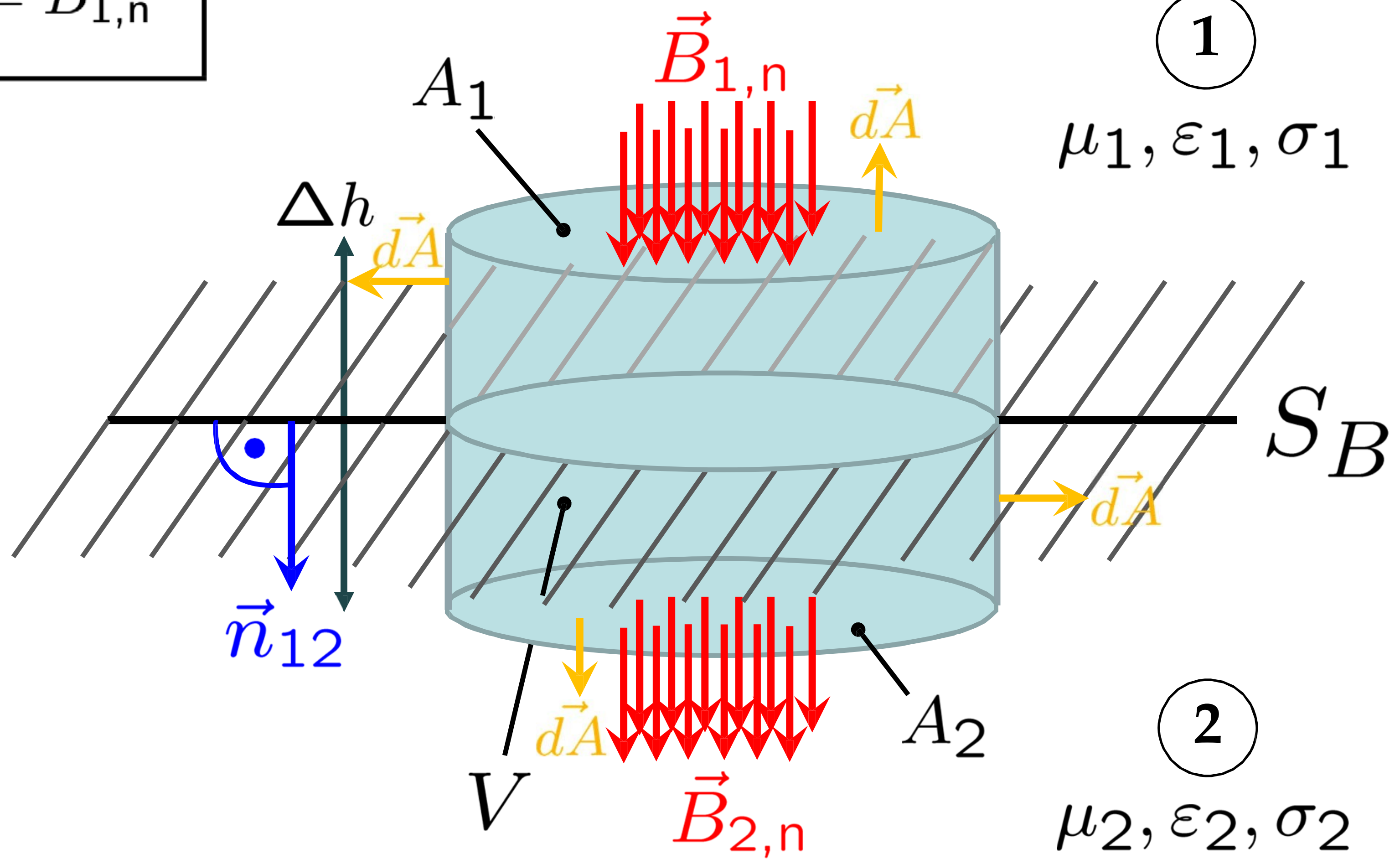
$$\vec{H}_{2,\tan} - \vec{H}_{1,\tan} = \vec{J}_S \times \vec{n}_{12}$$

$$\begin{aligned}\vec{H}_{1,\tan} &= \vec{n}_{12} \times \vec{H}_1 \times \vec{n}_{12} \\ \vec{H}_{2,\tan} &= \vec{n}_{12} \times \vec{H}_2 \times \vec{n}_{12}\end{aligned}$$

$$\boxed{\vec{n}_{12} \times \left(\vec{H}_2 - \vec{H}_1\right) = \vec{J}_S}$$

## Boundary Conditions

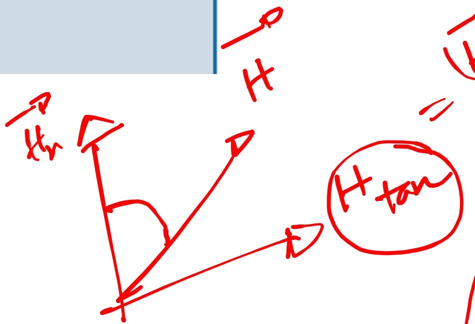
$$\boxed{B_{2,\mathrm{n}} = B_{1,\mathrm{n}}}$$



**PRACTICE EXERCISE 8.9**

A unit normal vector from region 2 ($\mu = 2\mu_o$) to region 1 ($\mu = \mu_o$) is $\mathbf{a}_{n21} = (6\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z)/7$. If $\mathbf{H}_1 = 10\mathbf{a}_x + \mathbf{a}_y + 12\mathbf{a}_z$ A/m and $\mathbf{H}_2 = H_{2x}\mathbf{a}_x - 5\mathbf{a}_y + 4\mathbf{a}_z$ A/m, determine

(a) $\mathbf{H}_{2x}$
(b) The surface current density $\mathbf{K}$ on the interface
(c) The angles $\mathbf{B}_1$ and $\mathbf{B}_2$ make with the normal to the interface




**P.E. 8.9**

(a) $\vec{B}_{1n} = \vec{B}_{2n} \rightarrow \mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n}$

or $\mu_1 \vec{H}_1 \bullet \vec{a}_{n21} = \mu_2 \vec{H}_2 \bullet \vec{a}_{n21}$

$$\mu_o \frac{(60+2-36)}{7} = 2\mu_o \frac{(6H_{2x}+10-12)}{7}$$

$35 = 6H_{2x}$

$\underline{\underline{H_{2x} = 5.833}}$

(b) $\vec{K} = (\vec{H}_1 - \vec{H}_2)\times \vec{a}_{n12} = \vec{a}_{n21}\times(\vec{H}_1 - \vec{H}_2)$

$= \vec{a}_{n21} \times \left[(1\ ,1,12) - (35/6, -5, 4)\right]$

$$= \frac{1}{7}\begin{vmatrix} 6 & 2 & -3 \\ 25/6 & 6 & 8 \end{vmatrix}$$

$\vec{K} = \underline{\underline{4.86\vec{a}_x - 8.64\vec{a}_y + 3.95\vec{a}_z}}$ A/m

(c) Since $\vec{B} = \mu\vec{H}$, $\vec{B}_1$ and $\vec{H}_1$ are parallel, i.e. they make the same angle with the normal to the interface.

$$\cos\theta_1 = \frac{\vec{H}_1 \bullet \vec{a}_{n21}}{|\vec{H}_1|} = \frac{26}{7\sqrt{100+1+144}} = 0.2373$$

$\underline{\underline{\theta_1 = 76.27^o}}$

$$\cos\theta_2 = \frac{\vec{H}_2 \bullet \vec{a}_{n21}}{|\vec{H}_2|} = \frac{13}{7\sqrt{(5.833)^2+25+16}} = 0.2144$$

$\underline{\underline{\theta_2 = 77.62^o}}$

## 3.3 Problem 3

The three dimensional space is split into two regions filled with different materials (see Fig. 3.3.2). The electric vector field $\vec{E}_1$ and the magnetic vector field $\vec{H}_1$, in region 1 (see Fig. 3.3.2), are given as

$$\vec{E}_1 = (2.0\vec{e}_y + 3.0\vec{e}_z)\,\mathrm{V/m} \qquad \vec{H}_1 = (0.1\vec{e}_x + 0.2\vec{e}_z)\,\mathrm{A/m}$$

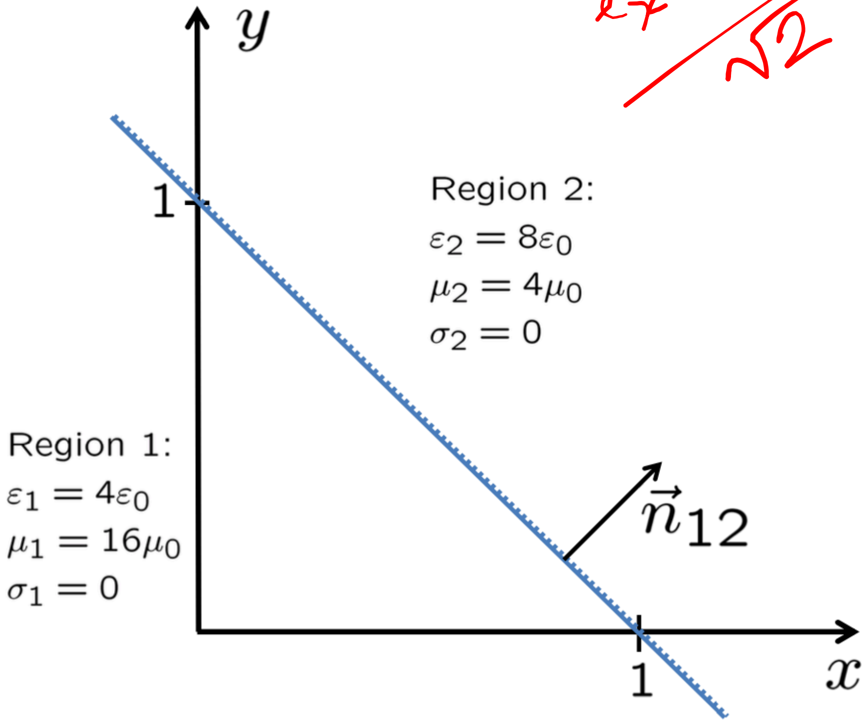
For sub-problems a) and b) there are no surface charges $\varrho_s$ on the interface.

a) Find the electric vector field $\vec{E}_2$ in region 2 so that the boundary conditions at the interface are fulfilled.

b) Find the magnetic induction $\vec{B}_2$ in region 2 so that the boundary conditions at the interface are fulfilled.

c) Now the medium in region 2 is replaced by an ideal conductor with conductivity of $\sigma_2 \to \infty$ while all other parameters remain the same. Determine the surface charge density $\varrho_S$ on the interface and determine the surface current density $\vec{J}_S$.

$\frac{\hat{e}_x + \hat{e}_y}{\sqrt{2}}$




3.4 b) $\vec H_{2tan} = \vec H_{1tan}$ because $\vec J_s = 0$

$H_1 - H_{1tan}$

$H_n$

$$\vec H_{1tan} = \vec n_{12} \times \vec H_1 \times \vec n_{12} = \frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0.1\\0\\0.2\end{pmatrix} \times \frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\end{pmatrix}\frac{A}{m}$$

$$= \frac{1}{\sqrt2}\begin{pmatrix}0.2\\-0.2\\-0.1\end{pmatrix} \times \frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\end{pmatrix}\frac{A}{m} = \frac{1}{2}\begin{pmatrix}0.1\\-0.1\\0.4\end{pmatrix}\frac{A}{m}$$

$$= \vec H_{2tan} \Rightarrow \vec B_{2tan} = \mu_2 \vec H_{2tan}$$

$$B_{2n} = B_{1n} \Rightarrow \mu_2 H_{2n} = \mu_1 H_{1n} \Rightarrow H_{2n} = \frac{\mu_1}{\mu_2} H_{1n} = 4 H_{1n}$$

$$H_{1n} = \vec n_{12} \cdot \vec H_1 = \frac{1}{\sqrt2}\begin{pmatrix}0.1\\0\\0.2\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix}\frac{A}{m} = \frac{0.1}{\sqrt2}\frac{A}{m}$$

$$H_{2n} = 4 H_{1n} = \frac{0.4}{\sqrt2}\frac{A}{m} \Rightarrow \vec H_{2n} = \frac{0.4}{\sqrt2}\underbrace{\frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\end{pmatrix}}_{\vec n_{12}}\frac{A}{m} = \begin{pmatrix}0.2\\0.2\\0\end{pmatrix}\frac{A}{m}$$

M

$$\vec B_2 = \mu_2 \vec H_2 = \mu_2\left(\vec H_{2tan} + \vec H_{2n}\right)$$

$$= \mu_2\left[\frac{1}{2}\begin{pmatrix}0.1\\-0.1\\0.4\end{pmatrix} + \begin{pmatrix}0.2\\0.2\\0\end{pmatrix}\right]\frac{A}{m} = \mu_2\begin{pmatrix}0.25\\0.15\\0.2\end{pmatrix}\frac{A}{m}$$

### 3.3 Problem 3

The three dimensional space is split into two regions filled with different materials (see Fig. 3.3.2). The electric vector field $\vec{E}_1$ and the magnetic vector field $\vec{H}_1$, in region 1 (see Fig. 3.3.2), are given as

$$\vec{E}_1 = (2.0\vec{e}_y + 3.0\vec{e}_z)\,\mathrm{V/m} \qquad \vec{H}_1 = (0.1\vec{e}_x + 0.2\vec{e}_z)\,\mathrm{A/m}$$

For sub-problems a) and b) there are no surface charges $\varrho_s$ on the interface.

a) Find the electric vector field $\vec{E}_2$ in region 2 so that the boundary conditions at the interface are fulfilled.

b) Find the magnetic induction $\vec{B}_2$ in region 2 so that the boundary conditions at the interface are fulfilled.

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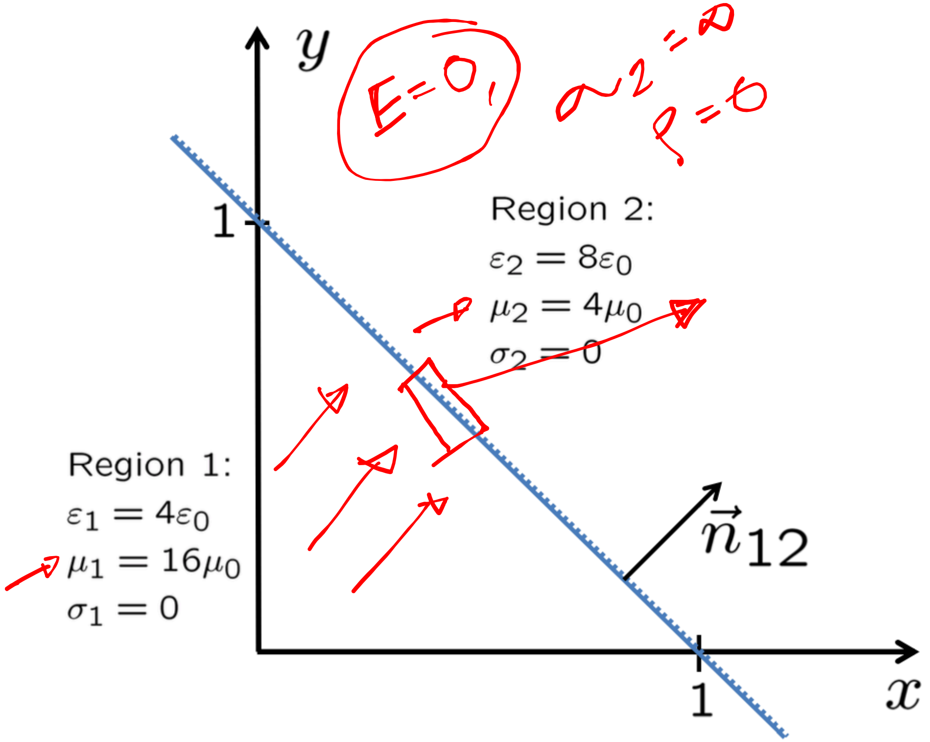
c) Now the medium in region 2 is replaced by an ideal conductor with conductivity of $\sigma_2 \to \infty$ while all other parameters remain the same. Determine the surface charge density $\varrho_S$ on the interface and determine the surface current density $\vec{J}_S$.




3.4c)

$$\vec{J}_S = \vec{n}_{12} \times (\vec{H}_2 - \vec{H}_1) = -\vec{n}_{12} \times \vec{H}_1 = \vec{H}_1 \times \vec{n}_{12}$$

$$= \begin{pmatrix} 0.1 \\ 0 \\ 0.2 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \frac{A}{m} = \begin{pmatrix} -\frac{0.2}{\sqrt{2}} \\ \frac{0.2}{\sqrt{2}} \\ \frac{0.1}{\sqrt{2}} \end{pmatrix} \frac{A}{m}$$

$$\cancel{D_{2n}}^{\,0} - D_{1n} = \varrho_S \Rightarrow \varrho_S = -D_{1n} = -\varepsilon_1 E_{1n}$$

$$E_{1n} = \vec{n}_{12} \cdot \vec{E}_1 = \sqrt{2}\,\mathrm{V/m}$$

$$\Rightarrow \varrho_S = -\varepsilon_1 \sqrt{2}\,\frac{V}{m} \quad \text{with } \varepsilon_1 = 4\varepsilon_0 = 4 \cdot 8.854 \cdot 10^{-12} \frac{As}{Vm}$$

$$\Rightarrow [\varrho_S] = \frac{As}{m^2} = \frac{C}{m^2}$$

## *Maxwell's Equations*

$$B_{2,\mathrm{n}} = B_{1,\mathrm{n}}$$

**Boundary conditions at a perfect electric conductor:**

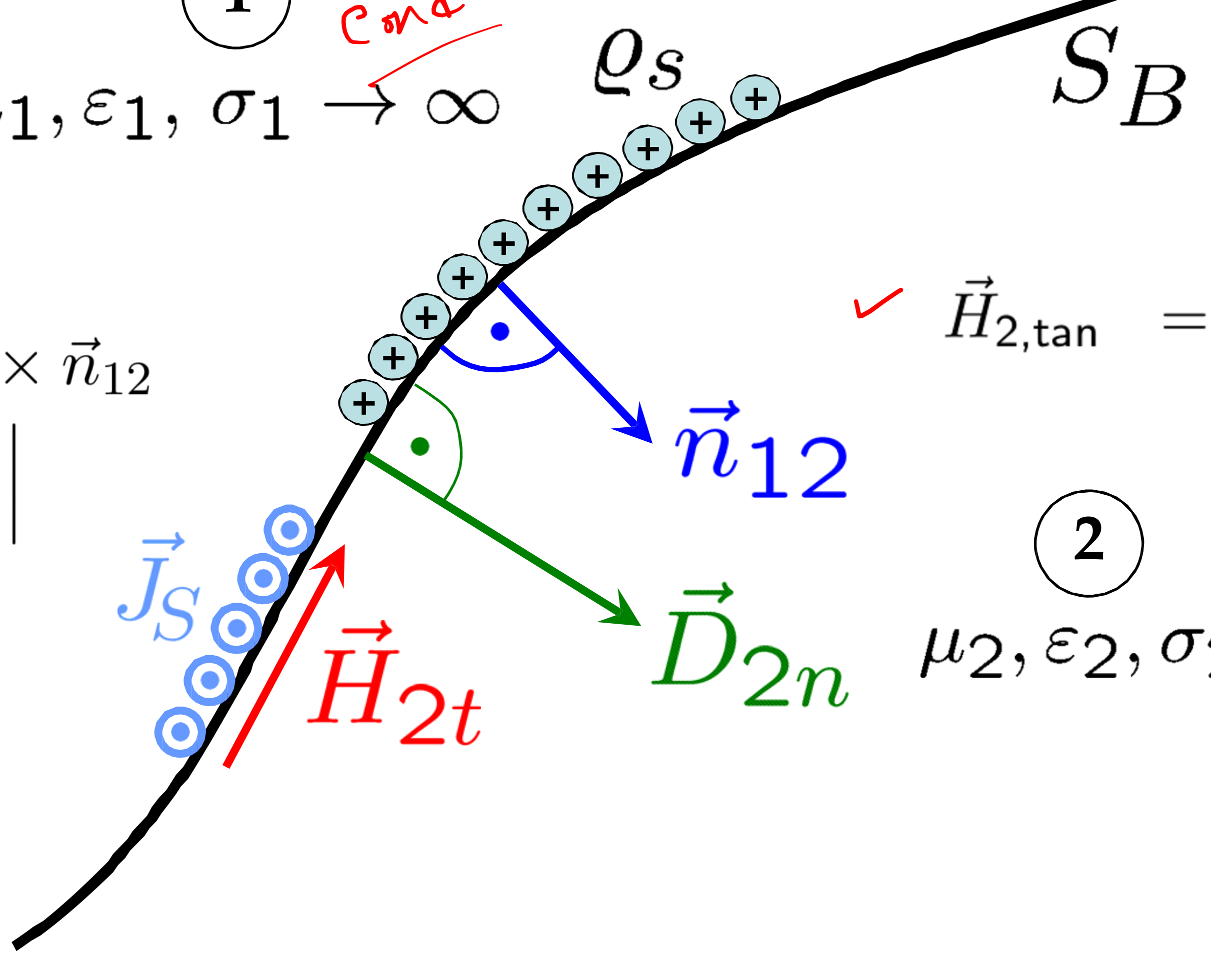
$$\vec{n}_{12} \times \left(\vec{H}_2 - \vec{H}_1\right) = \vec{J}_S$$

$\vec{E}=0$

$\vec{H}_1 = 0$

1

$\mu_1, \varepsilon_1, \sigma_1 \to \infty$ cond




$$\begin{aligned}
\vec{n}_{12} \times \vec{H}_2 &= \vec{J}_S \quad \times \vec{n}_{12} \\
\vec{H}_{2t} &= \vec{J}_S \times \vec{n}_{12} \\
\left|\vec{H}_{2t}\right| &= \left|\vec{J}_S\right|
\end{aligned}$$

$$\vec{H}_{2,\mathrm{tan}} = \vec{n}_{12} \times \vec{H}_2 \times \vec{n}_{12}$$

2

$\mu_2, \varepsilon_2, \sigma_2$

$$B_{2n} = 0$$

## Maxwell's Equations

**Boundary conditions at a perfect electric conductor:**

$\sigma_1 \to \infty$

$$
\begin{aligned}
\vec{n}_{12} \times \vec{H}_2 &= \vec{J}_S \\
\vec{H}_{2t} &= \vec{J}_S \times \vec{n}_{12} \\
\left|\vec{H}_{2t}\right| &= \left|\vec{J}_S\right| \\
\vec{E}_{2t} &= \vec{0} \\
D_{2n} &= \varrho_S \\
B_{2n} &= 0
\end{aligned}
$$

## 3.5 Problem 5

Two very large metallic plates with an area $A$ are placed next to each other in a distance $d$ (see Fig. 3.5.4).

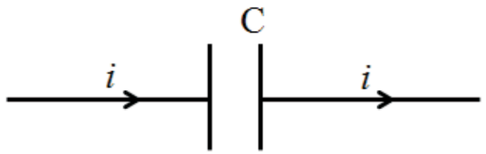



Figure 3.5.4

Derive the capacitance $C$ between the plates.

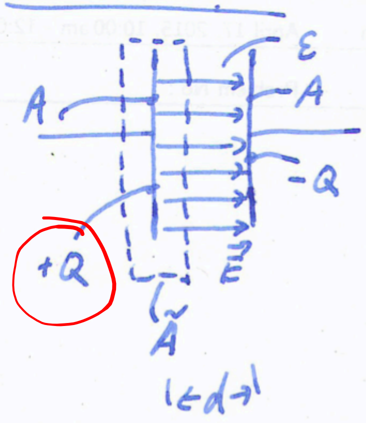
Hint: Assume that one plate carries positive and the other plate the same amount of negative charges. Additionally assume a pure homogeneous field (because the plates are very large).

$$C = \frac{\epsilon A}{d}$$

$V = E \times D$

Problem 3.5

(12)




$$Q = CV \Rightarrow C = \frac{Q}{V}$$

$$Q = ?\ ;\ V = ?$$

$$\oiint_A \vec{D}\cdot d\vec{A} = \iiint_V \rho \, dV = Q$$

here: $\vec{D} \parallel d\vec{A}$ and $\vec{D}$ is constant (homogeneous) between the plates and zero elsewhere.

$$\Rightarrow |\vec{D}|\cdot A = Q \ ;\ |\vec{D}| = \varepsilon \cdot |\vec{E}|$$

$$\Rightarrow Q = \varepsilon |\vec{E}| \cdot A$$

$$V = \int_{\text{plate 1}}^{\text{plate 2}} \vec{E}\cdot d\vec{s}$$ ; here: $\vec{E} \parallel d\vec{s}$ and $\vec{E}$ is constant (homogeneous) between the plates.

$$\Rightarrow |\vec{E}|\cdot d = V$$

$\Rightarrow$ Capacitance $C = \frac{Q}{V} = \frac{\varepsilon |\cancel{\vec{E}}| A}{|\cancel{\vec{E}}| d}$

$$= \varepsilon \frac{A}{d}$$