Lecture 3

Continuity Theorem, Maxwell’s 4th Equations, Poynting Vector

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2.3 Problem 3

The vector field

$$\vec{F} = \begin{pmatrix} ax^2 \\ byz \\ x \end{pmatrix}$$

is given in Cartesian coordinates with the constants a and b.

a) Determine the divergence of the vector field $\vec{F}$.

b) Verify Gauss' theorem

$$\iiint\limits_V \operatorname{div} \vec{F} \, \mathrm{d}V = \oiint\limits_A \vec{F} \cdot \mathrm{d}\vec{A}$$

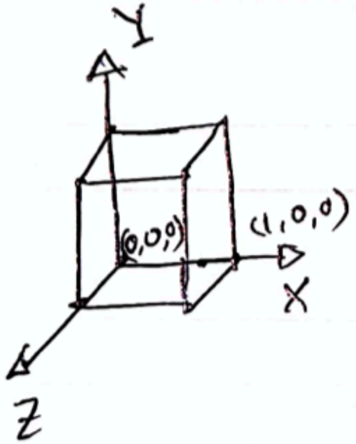
for a cube which fills the space between the points $(0,0,0)$ and $(1,1,1)$.

2.3

a) $$\nabla \cdot F = \frac{\partial}{\partial x}(F_x) + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$= 2ax + bz + 0$$

ⓑ $$L.H.S. = \iiint\limits_V \operatorname{div} \vec{F} \, dV$$



$$= \int_0^1\int_0^1\int_0^1 (2ax + bz)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 [ax^2 + bzx]_0^1 \, dy\, dz$$

$$= \int_0^1\int_0^1 (a + bz)\, dy\, dz$$

$$= \int_0^1 [ay + byz]_0^1 \, dz$$

$$= \int_0^1 (a + bz)\, dz = [az + bz^2/2]_0^1$$

$$= a + b/2$$

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$$\text{R.H.S.} = \oiint\limits_A \vec{F} \cdot d\vec{A}$$

Total 6 surface in the cubes $\Rightarrow$

For 2 XY planes

(i) $$\int_{y=0}^{1}\int_{x=0}^{1} (ax^2\hat{i} + byz\hat{j} + x\hat{k}) \cdot (dx\,dy\,\hat{k}) \Big|_{z=0}$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} x\,dx\,dy$$

$$= \int_{y=0}^{1} \left[\frac{x^2}{2}\right]_0^1 dy$$

$$= 1/2$$

(ii) $$\int_{y=0}^{1}\int_{x=0}^{1} (ax^2\hat{i} + byz\hat{j} + x\hat{k}) \cdot [dx \cdot dy(-\hat{k})] \Big|_{z=1}$$

$$= -\int_{y=0}^{1}\int_{x=0}^{1} x\,dx\,dy$$

$$= -1/2$$

So, for XY plane $\oiint\limits_A \vec{F} \cdot d\vec{A} = 1/2 - 1/2$
$= 0$

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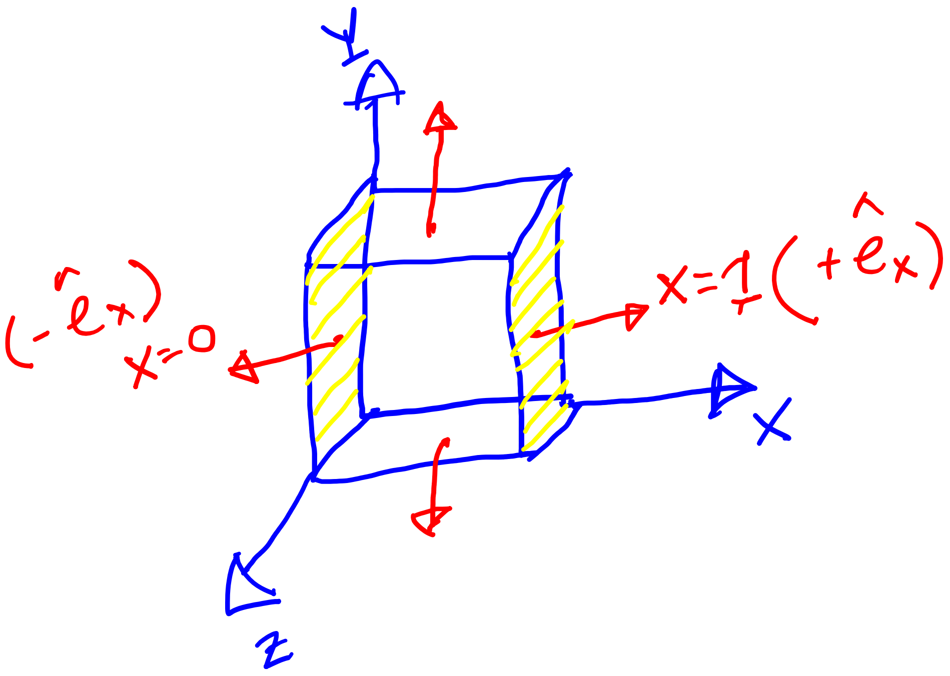


For YZ plane

$$\int_{z=0}^{1}\int_{y=0}^{1} ax^2\,(\cancel{+}\,dy\,dz)\Big|_{x=0\,(\cancel{+})\,(-),\ x=1\,(\cancel{-})\,+}$$

$= a$

[** surface integral always outward**]

For XZ plane

$$\int_{z=0}^{1}\int_{x=0}^{1} byz\,(\cancel{+}\,dx\,dz)\Big|_{y=0\,(\cancel{+})\,(-),\ y=1\,(\cancel{-})\,(+)}$$

$= \int_{z=0}^{1} bz\,dz$

$= b\left[\frac{z^2}{2}\right]_0^1$

$= \frac{b}{2}$

So, total

$\oiint \vec{F}\cdot d\vec{A} = a + b/2$

L.H.S. = R.H.S.

Previous Day's Equations

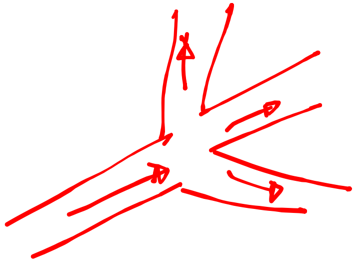
TABLE 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampère's circuit law

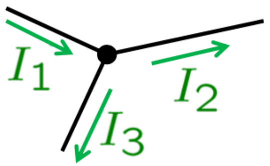
*This is also referred to as Gauss's law for magnetic fields.

Continuity Equation

Continuity Equation: When a fluid is in motion, it must move in such a way that mass is conserved.



Correponds to Kirchhoff's Current Law (KCL):



$$-I_1 + I_2 + I_3 = 0$$

$$I_1 = I_2 + I_3$$

From the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the surface of the volume. Thus current I_{out} coming out of the closed surface is

Qin I J

$$I_{\text{out}} = \oint_S \mathbf{J} \cdot d\mathbf{S} = \frac{-dQ_{\text{in}}}{dt} \tag{5.40}$$

where Q_{in} is the total charge enclosed by the closed surface. Invoking the divergence theorem, we write

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{J}\, dv \tag{5.41}$$

But

$$\frac{-dQ_{\text{in}}}{dt} = -\frac{d}{dt}\int_v \rho_v\, dv = -\int_v \frac{\partial \rho_v}{\partial t}\, dv \tag{5.42}$$

Substituting eqs. (5.41) and (5.42) into eq. (5.40) gives

$$\int_v \nabla \cdot \mathbf{J}\, dv = -\int_v \frac{\partial \rho_v}{\partial t}\, dv$$

$\cancel{\frac{\partial \rho_v}{\partial t} = 0}$ DC Current

$\nabla \cdot J = 0$

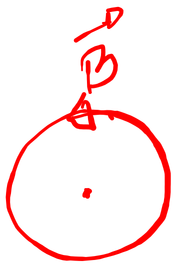
$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}} \tag{5.43}$$

Ampere’s Law

Ampère's circuital law relates the integrated magnetic field around a closed loop is proportionate to the electric current passing through the loop.

$2\pi r$

$\vec{H} = \frac{\vec{B}}{\mu_0}$

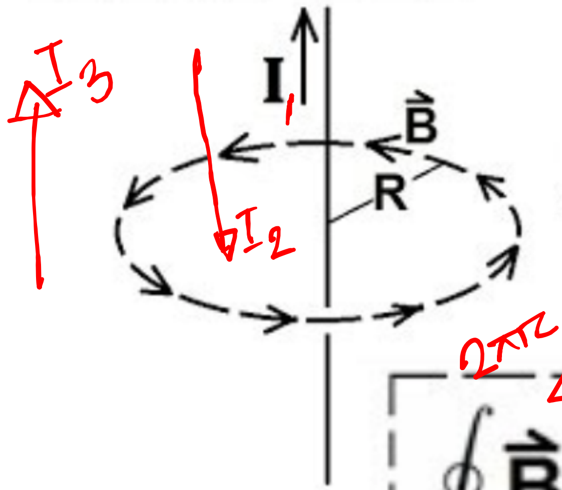


$$I_{enc} = \oint \mathbf{H} \cdot d\mathbf{L} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

$$I_{enc} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$\Rightarrow \quad \nabla \times \mathbf{H} = \mathbf{J}$$



closed Loop integral

$2\pi r$

$$\boxed{\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{encl}}$$

0

$(I_1 - I_2)$

Maxwell's Fourth Equation (Toughest One)

$\nabla \times H$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \tag{9.17}$$

But the divergence of the curl of any vector field is identically zero (see Example 3.10). Hence,

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} \tag{9.18}$$

The continuity of current in eq. (5.43), however, requires that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0 \tag{9.19}$$

Thus eqs. (9.18) and (9.19) are obviously incompatible for time-varying conditions. We must modify eq. (9.17) to agree with eq. (9.19). To do this, we add a term to eq. (9.17) so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \tag{9.20}$$

where $\mathbf{J}_d$ is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d \tag{9.21}$$

In order for eq. (9.21) to agree with eq. (9.19),

1st

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{9.22a}$$

or

$$\boxed{\mathbf{J}_d = \frac{\partial \mathbf{D}}{dt}} \tag{9.22b}$$

$\frac{\partial(\epsilon E)}{dt}$

Substituting eq. (9.22b) into eq. (9.20) results in

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \tag{9.23}$$

$\epsilon \frac{\partial E}{\partial t}$

This is Maxwell's equation (based on Ampère's circuit law) for a time-varying field. The term $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ is known as *displacement current density* and $\mathbf{J}$ is the conduction current

Displacement Current Density: Displacement current density is the quantity ∂D/∂t appearing in Maxwell's equations that is defined in terms of the rate of change of D, the electric displacement field.

$\frac{\epsilon A}{d}$

PRACTICE EXERCISE 9.4

In free space, $\mathbf{E} = 20\cos(\omega t - 50x)\,\mathbf{a}_y$ V/m. Calculate

(a) $\mathbf{J}_d$

(b) $\mathbf{H}$

(c) ω

$\omega t - 20z - 50x$

(a) $J_d = \dfrac{\partial D}{\partial t}$

$= \epsilon_0 \dfrac{\partial E}{\partial t}$

$= \epsilon_0 \omega \times 20 \times (-1) \times \sin(\omega t - 50x)\,\hat{a}_y$

$= -20\omega\epsilon_0 \sin(\omega t - 50x)\,\hat{a}_y$ A/m²

(b) $\nabla \times \vec{H} = \vec{J} + \vec{J}_D$

$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} = \vec{J}_D$

$\hat{a}_z\left[\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right]$

H_x X

H_y X

$\hat{a}_y\left(\cancel{\dfrac{\partial H_x}{\partial z}} - \dfrac{\partial H_z}{\partial x}\right) = \vec{J}_D$

$\hat{a}_y$ $\hat{a}_y$

$-\dfrac{\partial H_z}{\partial x}\,\hat{a}_y = \vec{J}_D$

$\vec{H}_z = -\int \vec{J}_D \cdot dx$

$\hat{a}_z$

$= \dfrac{20}{50}\omega\epsilon_0 \cos(\omega t - 50x)\,\hat{a}_z$

$= 0.4\,\omega\epsilon_0 \cos(\omega t - 50x)\,\hat{a}_z$ A/m

(c)

$$\nabla \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t} \rightarrow -\frac{\partial E_y}{\partial x}\vec{a}_z = 0.4\mu_o w \varepsilon_o \sin(wt - 50x)\vec{a}_z$$

$\hat{a}_z$

$$1000 = 0.4\mu_o\varepsilon_o w^2 = 0.4\frac{w^2}{c^2}$$

$\Rightarrow \omega = \sqrt{\dfrac{1000 \times c^2}{0.4}}$

or $w = \underline{1.5 \times 10^{10}}$ rad/s

3.1 Problem 1

Use Gauss' and Stokes' theorems to transform the Maxwell equations from differential to integral notation.

$$\mathrm{curl}\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\mathrm{curl}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\mathrm{div}\vec{D} = \varrho$$
$$\mathrm{div}\vec{B} = 0$$

3.1 (a) $\mathrm{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$$\iint_A (\mathrm{Curl}\,\vec{H})\cdot dA = \iint_A \vec{J}\cdot d\vec{A} + \frac{\partial}{\partial t}\iint_A \vec{D}\cdot d\vec{A}$$

$$\oint_L H\cdot dl = \iint_A \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{A}$$

Stoke's theorem

(b) $\mathrm{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\iint_A (\mathrm{Curl}\,\vec{E})\cdot dA = -\frac{\partial}{\partial t}\iint_A \vec{B}\cdot d\vec{A}$$

$$\oint_L \vec{E}\cdot dl = -\frac{\partial}{\partial t}\iint_A \vec{B}\cdot d\vec{A}$$

$\int A\cdot dl = \iint \nabla\times A\, dA$

$\iint A\cdot dA = \iiint \nabla\cdot A\, dv$

(c) $\mathrm{div}\,\vec{D} = \rho$

$$\iiint \mathrm{div}\,\vec{D}\, dv = \iiint \rho\, dv$$

$$\oiint_A \vec{D}\cdot d\vec{A} = Q$$

(d) $\mathrm{div}\,\vec{B} = 0$

$$\iiint \mathrm{div}\,\vec{B}\, dv = 0$$

$$\oiint_A \vec{B}\cdot d\vec{A} = 0$$

3.2 Problem 2

According to Faraday's law of electromagnetic induction, a time varying magnetic flux ψ_m penetrating a loop induces an electromotive force (emf) noted as voltage $v(t)$ such that

$$v(t) = -\frac{d\psi_m}{dt} \qquad (3.2.1)$$

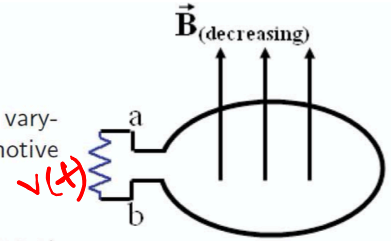


Figure 3.2.1

a) Give Faraday's law in point form.

b) Express eq. (3.2.1) in integral form, in terms of $\vec{E}$ and $\vec{B}$ fields.

c) Determine the polarities of terminals a and b in the loop shown in Fig. 3.2.1. The $\vec{B}$ field shall be continuously decreasing over time.

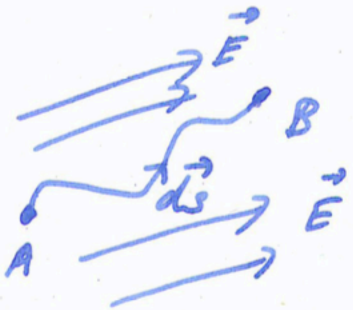
$\boxed{V = E \times d}$

Problem 3.2 (8)

a) $\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

b) $v(t) = -\frac{d\Psi_m}{dt}$ $\qquad \Psi_m :=$ Magnetic Flux

$v_{AB}(t) = \int_A^B \vec{E}\cdot d\vec{s}$



$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Big| \iint_A \ldots \cdot d\vec{A}$

$\iint_A (\text{curl}\,\vec{E})\cdot d\vec{A} = -\iint_A \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$

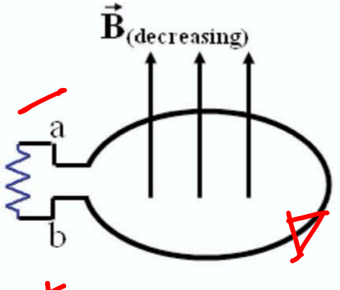
"Stokes" $\qquad = -\frac{\partial}{\partial t}\underbrace{\iint_A \vec{B}\cdot d\vec{A}}_{\Psi_m(t)}$

$\oint \vec{E}\cdot d\vec{s} = -\frac{\partial}{\partial t}\iint_A \vec{B}\cdot d\vec{A}$

$\Psi_m(t) = L\cdot I(t)$

$v(t) = -L\frac{di(t)}{dt}$

3.2 Problem 2



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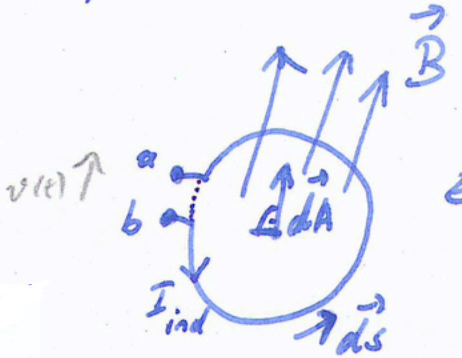
c) $\vec{ds}$ and $\vec{dA}$ are right-handed oriented.



← here, $\vec{B}$ and $\vec{dA}$ are in parallel. Then, because $\vec{B}$ itself is decreasing, $\iint_A \vec{B}\cdot\vec{dA}$ is decreasing over time. Therefore, the rate of change of $\iint_A \vec{B}\cdot\vec{dA}$ is negative. Together with the minus-sign in Faraday's Law the integral $\oint \vec{E}\cdot\vec{ds}$ is positive, if we integrate from terminal b to terminal a. $\Rightarrow b \rightarrow \oplus$; $a \rightarrow \ominus$.

$$\oint \vec{E}\cdot\vec{ds} = -\frac{\partial}{\partial t}\iint_A \vec{B}\cdot\vec{dA}$$

(−) (−)

(+)

Poynting Theorem

Poynting's theorem states that the net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within v minus the ohmic losses.

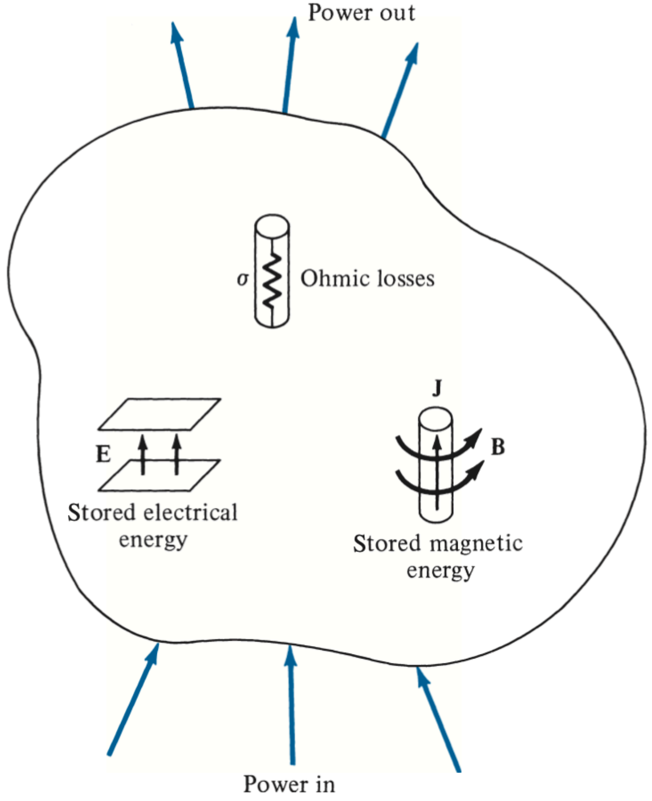


FIGURE 10.11 Illustration of power balance for EM fields.

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_v \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dv - \int_v \sigma E^2 dv$$

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ \text{total power} & & \text{rate of decrease in} & & \text{ohmic power} \\ \text{leaving the volume} & = & \text{energy stored in electric} & - & \text{dissipated} \\ & & \text{and magnetic fields} & & \end{array}$$

HF

Poynting's Theorem:

Power dissipation into heat

$$\oint\!\!\!\oint_A \left(\vec{E}\times\vec{H}\right)\cdot d\vec{A} + \iiint_V \sigma\left(\vec{E}\cdot\vec{E}\right)dV =$$

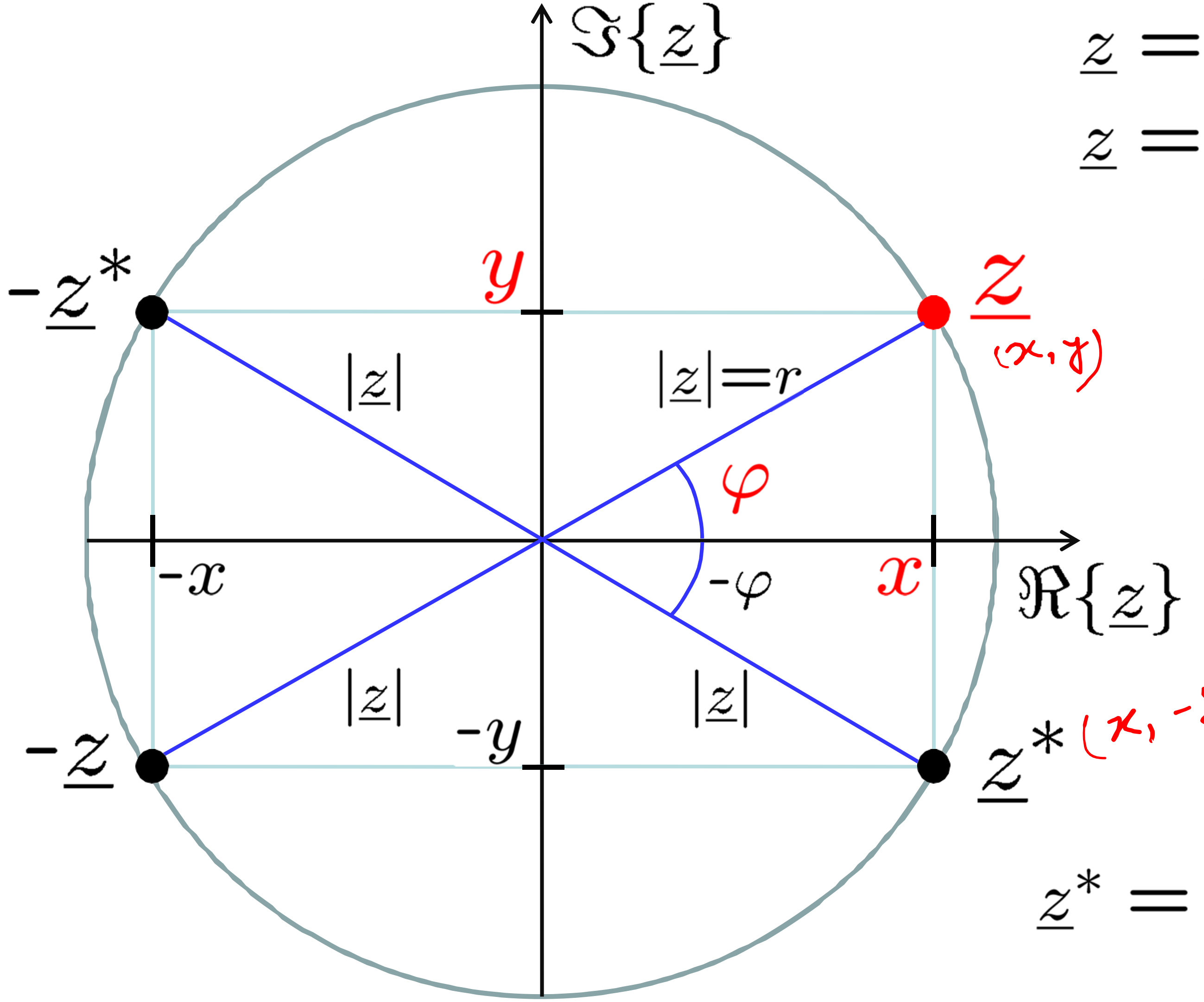
$$\left[-\frac{\partial}{\partial t}\iiint_V \frac{\varepsilon}{2}\left(\vec{E}\cdot\vec{E}\right)dV - \frac{\partial}{\partial t}\iiint_V \frac{\mu}{2}\left(\vec{H}\cdot\vec{H}\right)dV\right]$$

Power Flow
(out of volume *V*)

Energy of the Electric Field

Energy of the Magnetic Field

Complex Numbers and Conjugate Complex



$$\underline{z} = r\,e^{j\varphi}$$

$$\underline{z} = x + jy$$

$x - jy$

$\cancel{x} = \frac{1}{2}(\underline{z} + \underline{z}^*)$

$\boxed{Re(\underline{z}) = 1/2(\underline{z} + \underline{z}^*)}$

$$\underline{z}^* = x - jy$$

Maxwell's Equations

Poynting's Theorem:

$$\boxed{\oint\!\!\!\oint_A \left(\vec{E} \times \vec{H}\right) \cdot d\vec{A} \quad + P_V = -\frac{\partial}{\partial t}\left(W_{\text{el}} + W_{\text{magn}}\right)}$$

Energy flow

Poynting vector $\vec{S}$:

$$\boxed{\vec{S} = \vec{E} \times \vec{H}}$$

$$\frac{J}{s\,m^2}$$

Poynting vector $\cancel{\mathfrak{P}}$ $\vec{S}$, measured in watts per square meter (W/m^2);

Maxwell's Equations

Poynting Vector: $\vec{S}(\omega t) = \vec{E}(\omega t) \times \vec{H}(\omega t)$

$\mathbb{P} = \text{Real} + \cancel{\text{Complex}}$

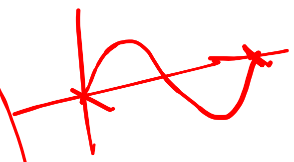
Real

average power

$$\begin{aligned}
\vec{S} &= \vec{E} \times \vec{H} \\
&= \Re\left\{\underline{\vec{E}}\,\mathrm{e}^{j\omega t}\right\} \times \Re\left\{\underline{\vec{H}}\,\mathrm{e}^{j\omega t}\right\} \\
&= \frac{1}{2}\left(\underline{\vec{E}}\,\mathrm{e}^{j\omega t} + \underline{\vec{E}}^{*}\,\mathrm{e}^{-j\omega t}\right) \times \frac{1}{2}\left(\underline{\vec{H}}\,\mathrm{e}^{j\omega t} + \underline{\vec{H}}^{*}\,\mathrm{e}^{-j\omega t}\right) \\
&= \frac{1}{4}\left(\underline{\vec{E}} \times \underline{\vec{H}}^{*} + \underline{\vec{E}}^{*} \times \underline{\vec{H}} + \underline{\vec{E}} \times \underline{\vec{H}}\mathrm{e}^{j2\omega t} + \underline{\vec{E}}^{*} \times \underline{\vec{H}}^{*}\mathrm{e}^{-j2\omega t}\right) \\
&= \frac{1}{4}\left(\underline{\vec{E}} \times \underline{\vec{H}}^{*} + \left(\underline{\vec{E}} \times \underline{\vec{H}}^{*}\right)^{*} + \underline{\vec{E}} \times \underline{\vec{H}}\mathrm{e}^{j2\omega t} + \left(\underline{\vec{E}} \times \underline{\vec{H}}\mathrm{e}^{j2\omega t}\right)^{*}\right) \\
&= \frac{1}{2}\Re\left\{\underline{\vec{E}} \times \underline{\vec{H}}^{*}\right\} + \frac{1}{2}\Re\left\{\underline{\vec{E}} \times \underline{\vec{H}}\,\mathrm{e}^{j2\omega t}\right\}
\end{aligned}$$

Maxwell's Equations

$$\vec{S}(t) = \frac{1}{2}\Re\left\{\underline{\vec{E}} \times \underline{\vec{H}}^*\right\} + \frac{1}{2}\Re\left\{\underline{\vec{E}} \times \underline{\vec{H}}\, e^{j2\omega t}\right\}$$



Complex Poynting Vector:

$$\boxed{\underline{\vec{S}} = \frac{1}{2}\,\underline{\vec{E}} \times \underline{\vec{H}}^*}$$

The real part of $\underline{\vec{S}}$ equals the mean active power flow density!!

$$\overline{\vec{S}} = \Re\{\underline{\vec{S}}\}$$

Asking for the average over time of the power flow density, we see from eq. (3.59) that only the first term contributes because the first term is not time-dependent whereas the second term is a mean-free AC field varying with angular frequency of 2ω. The Poynting vector averaged over one time period T ($T = \frac{2\pi}{\omega}$) is therefore:

$\vec{S}$ = Real + Img

$\frac{1}{2} R\{\ \} + \frac{1}{2} I\{\ \}$

Avg power

$$\overline{\vec{S}} = \frac{1}{T}\int_0^T \vec{S}(t)\,dt = \frac{1}{2}\Re\left\{\underline{\vec{E}} \times \underline{\vec{H}}^*\right\} \quad (3.60)$$

Please note: $\vec{S}(\omega t) \neq \Re\{\underline{\vec{S}} \cdot e^{j\omega t}\}$

instead:

$$\begin{aligned}\vec{S}(\omega t) &= \vec{E}(\omega t) \times \vec{H}(\omega t) \\ &= \frac{1}{2}\Re\left\{\underline{\vec{E}} \times \underline{\vec{H}}^*\right\} + \frac{1}{2}\Re\left\{\underline{\vec{E}} \times \underline{\vec{H}}\, e^{j2\omega t}\right\}\end{aligned}$$

and:

$$\underline{\vec{S}} = \frac{1}{2}\,\underline{\vec{E}} \times \underline{\vec{H}}^*$$

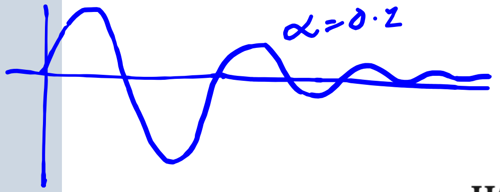
PRACTICE EXERCISE 10.8

In free space, $\mathbf{H} = 0.2\cos(\omega t - \beta x)\mathbf{a}_z$ A/m. Find the total power passing through:

(a) A square plate of side 10 cm on plane $x + y = 1$

(b) A circular disk of radius 5 cm on plane $x = 1$.

$\frac{1}{2} E \times H^*$ $\quad \vec{E}\ a_y$ $\quad f$



$$\mathbf{E}(z, t) = E_o e^{-\alpha z}\cos(\omega t - \beta z)\mathbf{a}_x$$

H_0

$$\mathbf{H}(z, t) = \frac{E_o}{|\eta|} e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\mathbf{a}_y$$

$\eta = 120\pi$

$\vec{P}(\omega t)$ $\quad \vec{P}_{avg}$

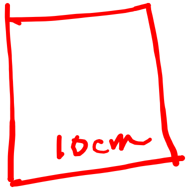
$\frac{E_o^2}{|\eta|} \times \eta \rightarrow H_0^2 \eta$

$$\mathcal{P}(z, t) = \frac{E_o^2}{|\eta|} e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta_\eta)\mathbf{a}_z$$

$$= \frac{E_o^2}{2|\eta|} e^{-2\alpha z}[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)]\mathbf{a}_z$$

C $\quad$ D

$\mathbb{P}(z,t) = \frac{1}{2}\eta \times H_0^2\, \hat{a}_x$



(a) normal vector on the plane is

$\nabla f = \hat{e}_x \frac{\partial}{\partial x}(x) + \hat{e}_y \frac{\partial}{\partial y}(y)$

$= \hat{e}_x + \hat{e}_y$

unit normal vector $= d\hat{n} = \frac{\hat{e}_x + \hat{e}_y}{\sqrt{2}}$

$\frac{1}{T}\int_0^T \cos^2(\omega t - \beta t) = 1$

$\boxed{\mathbb{P}_{avg} = \int_s \mathbb{P}(z,t)\, ds}$

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\mathbb{P}_{avg} = \int_s \mathbb{P}(z,t)\, ds$

$= \mathbb{P}(z,t) \times S \times d\hat{n}$

$= \frac{1}{2} \times \eta \times H_0^2 \times S \times \left(\frac{\hat{e}_x + \hat{e}_y}{\sqrt{2}}\right)$

$= \frac{1}{2} \times 120\pi \times 0.2^2 \times (10 \times 10^{-2})^2 \times \frac{1}{\sqrt{2}}$

$= 53.314 \times 10^{-3}$ mW

PRACTICE EXERCISE 10.8

In free space, $\mathbf{H} = 0.2\cos(\omega t - \beta x)\mathbf{a}_z$ A/m. Find the total power passing through:

(a) A square plate of side 10 cm on plane $x + y = 1$
(b) A circular disk of radius 5 cm on plane $x = 1$.

$$\mathbf{E}(z, t) = E_o e^{-\alpha z}\cos(\omega t - \beta z)\mathbf{a}_x$$

$$\mathbf{H}(z, t) = \frac{E_o}{|\eta|} e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\mathbf{a}_y$$

$$\mathcal{P}(z, t) = \frac{E_o^2}{|\eta|} e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta_\eta)\mathbf{a}_z$$

$$= \frac{E_o^2}{2|\eta|} e^{-2\alpha z}[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)]\mathbf{a}_z$$

(b) $P_{avg} = \frac{1}{2} \times \eta \times H_o^2 \times \pi \times (5 \times 10^{-2})^2 \times \underline{1}$

$= 59.22\,mW$

$P_{avg} = \int_s P(z,t)\,dt$