Lecture 2

Mathematical Problems, Maxwell's Equations, Continuity Equation

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2.1 Problem 1

A bee sits on a lamp located at the point $(2\,\mathrm{m}, 1\,\mathrm{m}, 4\,\mathrm{m})$ in a room. The temperature in the room is given by

$$\Phi/{}^\circ\mathrm{C} = ax^3 + by^2 + cz$$

where a, b, and c are constants. In which direction should the bee fly in order to reach a colder region in the room as fast as possible?

2.1 $\phi/{}^\circ C = ax^3 + by^2 + cz$

$P \equiv (2m, 1m, 4m)$

$$\text{grad}\,\phi = \begin{bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{bmatrix} = \begin{bmatrix} 3ax^2 \\ 2by \\ c \end{bmatrix}$$

$x = 2m$

$$\phi_P = \begin{bmatrix} 12a \\ 2b \\ c \end{bmatrix}$$

The direction it is hottest $= 12a\,\hat{i} + 2b\,\hat{j} + c\,\hat{k}$

$\hat{e}_x$ $\hat{e}_y$ $\hat{e}_z$

So, the bee should fly in the opposite direction $= -12a\,\hat{i} + -2b\,\hat{j} + -c\hat{k}$

$\frac{\partial\phi}{\partial x}$ $\phi\ {}^\circ C$ $- \frac{\partial\phi}{\partial x}\hat{i}$

2.2 Problem 2

Two vectors

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

are given.

a) Determine the area of a parallelogram spanned by $\vec{a}$ and $\vec{b}$.

b) Determine the angle between $\vec{a}$ and $\vec{b}$.

2.2.

(a) $|\vec{a} \times \vec{b}| = \sqrt{75} = 5\sqrt{5}$

(b) $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$

$\theta = \cos^{-1}(-0.3273)$

$= 109.105^\circ$

2.4 Problem 4

A vector field $\vec{F} = \begin{pmatrix} x + 2y + az \\ bx - 3y - z \\ 4x + cy + 2z \end{pmatrix}$ is given in Cartesian coordinate system with the constants $a, b, c \in \Re$.

a) Determine the constants a, b and c, such that the vector field $\vec{F}$ is curl-free.

b) Using the constants from a), show that the vector field $\vec{F}$ is a gradient function of a scalar field $\varphi(x, y, z)$.

$\nabla \times \vec{F} = 0$

irrotational

2.4

(a) Curl $\vec{F} = 0$, for curl free

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \hat{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2)$$

So, for curl free,

$$\left\{ \begin{array}{l} a = 4 \\ b = 2 \\ c = -1 \end{array} \right.$$

2.4 Problem 4

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$\nabla \phi = F$

$\frac{\partial \phi_x}{\partial x} \neq F_x$

ϕ = Not given

$\frac{\partial \phi}{\partial x} \hat{i} = F_x \hat{i}$

$\phi_x = 2xy$

$\phi_y = 2xy$

ϕ

$\phi = 4xy$

$\phi_x \neq \phi_x = \cancel{4xy}$

2.4 (b) $\vec{F} = \begin{pmatrix} x+2y+4z \\ 2x-3y-z \\ 4x-y+2z \end{pmatrix}$ $a=4, b=2, c=-1$

Let's assume,

$\vec{F} = \text{grad}\, \phi(x,y,z)$

$\frac{\partial \phi}{\partial x} = x+2y+4z$

$\Rightarrow \phi_x = \int (x+2y+4z)\,dx$

$= \frac{x^2}{2} + 2xy + 4zx$

$\phi_y = 2xy - \frac{3}{2}y^2 - zy$

$\phi_z = 4xz - yz + z^2 + 2xy$

So, $\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3}{2}y^2 - yz + z^2$

$\frac{\partial \phi}{\partial x} = x+2y+4z$, ------ all are same

So, the vector field is a gradient function of a scalar field $\phi(x,y,z)$

2.5 Problem 5

a) Express the function $\Phi = \frac{A}{|\vec{r}|}$ in spherical and in Cartesian coordinates. A is constant. Compute $\vec{\nabla}\Phi$ in spherical and in Cartesian coordinates.

b) Determine the divergence of the vector field $\vec{G}$ with $\vec{G}(r,\vartheta,\varphi) = ar^3\vec{e}_r + br\,\sin\vartheta\vec{e}_\vartheta + \vartheta\vec{e}_\varphi$ (a and b are constants) in spherical coordinates.

c) Express the unit vectors $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ of a Cartesian coordinate system in spherical coordinates r, ϑ, φ. Hint: Use the gradient operator.

2.5

(a) $\phi = \frac{A}{|\vec{r}|}$

$\vec{r} = x\vec{x} + y\vec{y} + z\vec{z}$

Cartesian: -

$$\phi = \frac{A}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{-1/2}A\right] \quad nx^{n-1}$$

$$= -\tfrac{1}{2}A\,(x^2+y^2+z^2)^{-3/2}\times 2x$$

$$= -\frac{A}{|\vec{r}|^3}\times x$$

$$\frac{\partial\phi}{\partial y} = -\frac{A}{|\vec{r}|^3}\times y \qquad \frac{\partial\phi}{\partial z} = -\frac{A}{|\vec{r}|^3}\times z$$

$$\therefore \vec{\nabla}\phi = -\frac{A}{|\vec{r}|^3}\begin{pmatrix} x \\ y \\ z\end{pmatrix}\begin{array}{l}\hat{e}_x\\ \hat{e}_y\\ \hat{e}_z\end{array}$$

2.5

Spherical, $\phi = \frac{A}{r}$

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial r}\vec{e}_r + \cancel{\frac{1}{r}\frac{\partial\phi}{\partial\theta}}\vec{e}_\theta + \cancel{\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}}\vec{e}_\varphi$$

$$= -\frac{A}{r^2}\vec{e}_r$$

$$\vec{\nabla}\phi = -\frac{A}{r^2}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{array}{l}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_\phi\end{array}$$

2.5 Problem 5

a) Express the function $\Phi = \frac{A}{|\vec{r}|}$ in spherical and in Cartesian coordinates. A is constant. Compute $\vec{\nabla}\Phi$ in spherical and in Cartesian coordinates.

spherical θ

b) Determine the divergence of the vector field $\vec{G}$ with $\vec{G}(r,\vartheta,\varphi) = ar^3\vec{e}_r + br\,\sin\vartheta\vec{e}_\vartheta + \vartheta\vec{e}_\varphi$ (a and b are constants) in spherical coordinates.

c) Express the unit vectors $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ of a Cartesian coordinate system in spherical coordinates r, ϑ, φ. Hint: Use the gradient operator.

divergence Formula [spherical co-or]

2.5

(b) $\operatorname{div}\vec{G} = \vec{\nabla}\cdot\vec{G}$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 G_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, G_\theta) + \frac{1}{r\sin\theta}\left(\frac{\partial G_\phi}{\partial\phi}\right)$$

$br\sin^2\theta$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(ar^5) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(br\sin^2\theta) + \cancel{\frac{1}{r\sin\theta}\left(\frac{\partial\theta}{\partial\phi}\right)} \to 0$$

$$= 5ar^4 \times \frac{1}{r^2} + \frac{b\cancel{r}}{\cancel{r}\sin\cancel{\theta}} \times 2\sin\cancel{\theta}\cos\theta$$

$$= 5ar^2 + 2b\cos\theta$$

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$$\frac{\partial r_x}{\partial x} \rightarrow \frac{\partial x}{\partial x} \stackrel{!}{=} 1$$

(c) $\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$

$x = r\cos\phi\sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

Formula

$$\hat{e}_x = \text{grad}\, x = \text{grad}(r\sin\theta\cos\phi)$$

$$= \begin{pmatrix} \frac{\partial}{\partial r}(r\sin\theta\cos\phi) \\ \frac{1}{r}\frac{\partial}{\partial\theta}(r\sin\theta\cos\phi) \\ \frac{1}{\cancel{r}\cancel{\sin\theta}}\frac{\partial}{\partial\phi}(\cancel{r}\cancel{\sin\theta}\cos\phi) \end{pmatrix} \quad \begin{matrix} \hat{e}_r \\ e_\theta \\ e_\phi \end{matrix}$$

$$= \begin{pmatrix} \sin\theta\cos\phi \\ \cos\theta\cos\phi \\ \cancel{\cos}\,-\sin\phi \end{pmatrix}$$

$$\hat{e}_y = \text{grad}\, y = \begin{pmatrix} \sin\theta\sin\phi \\ \cos\theta\sin\phi \\ \cos\phi \end{pmatrix}$$

$$\hat{e}_z = \text{grad}\, z = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}$$

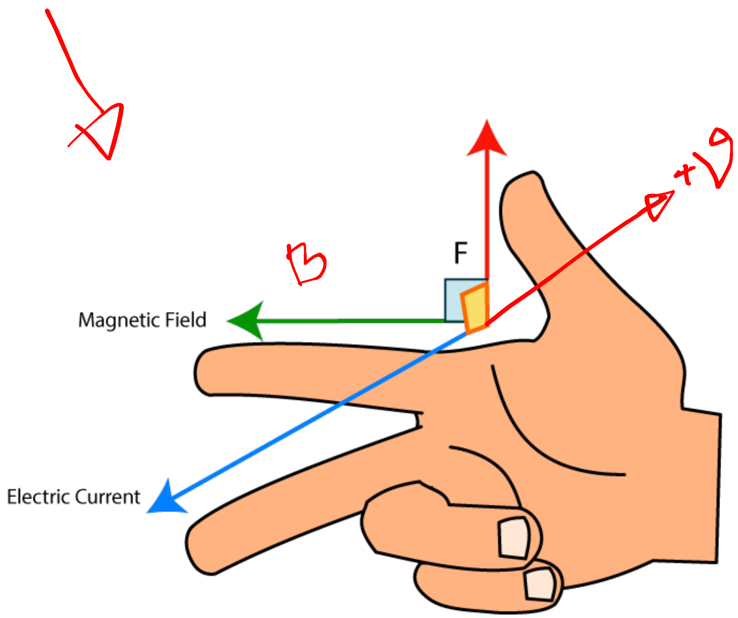
Thumb Rules

$\oint \vec{B} \cdot d\vec{s}$

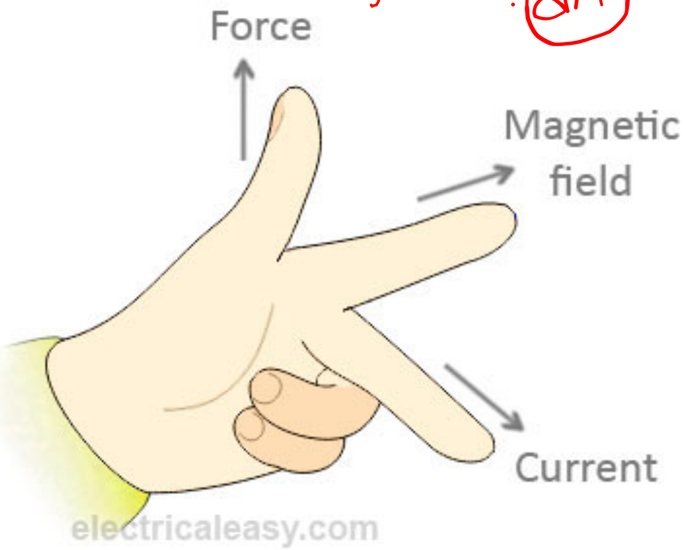
$d\vec{A}$

$I\, d\vec{l}$

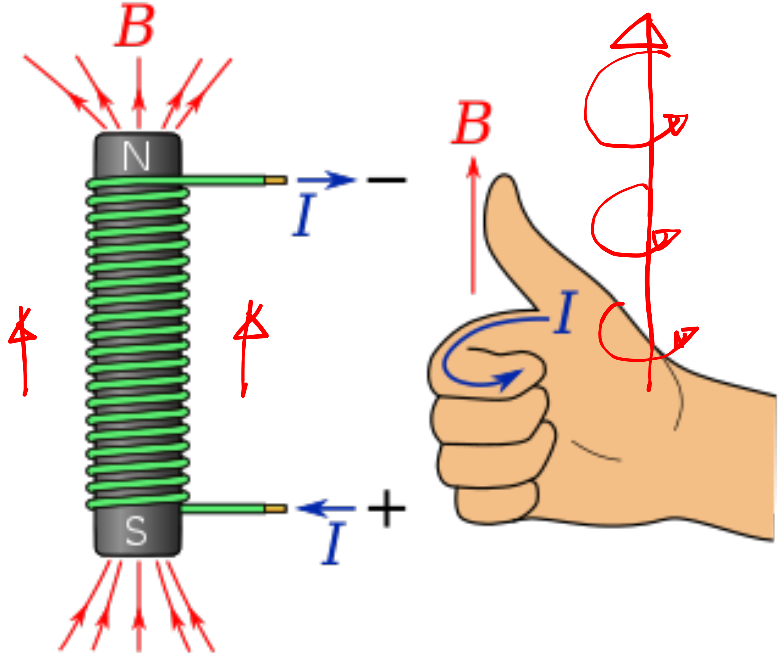
Line Inte



Fleming’s Right Hand Thumb Rule



Fleming’s Left Hand Thumb Rule



Thumb Rule for Magnetic Field

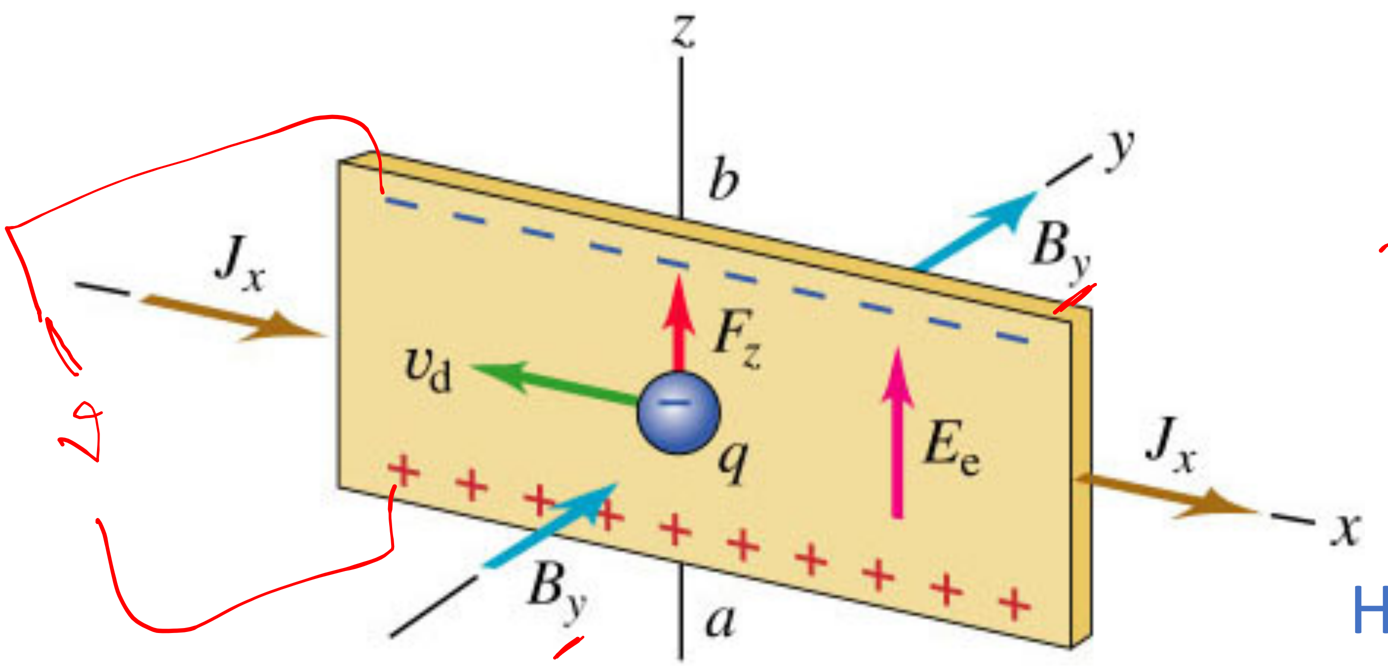
Maxwell's Equations

+1e

Electrodynamic Force (Lorentz Force) :

+q

$$\vec{F} = \underbrace{q\vec{E}}_{\text{Coulomb Force}} + \underbrace{q\,\vec{v}_q \times \vec{B}}_{\text{Magnetic Force}}$$



Hall Effect

Stoke's Theorem

Stokes's theorem states that the circulation of a vector field **A** around a (closed) path *L* is equal to the surface integral of the curl of **A** over the open surface *S* bounded by *L* (see Figure 3.21), provided **A** and $\nabla \times \mathbf{A}$ are continuous on *S*.

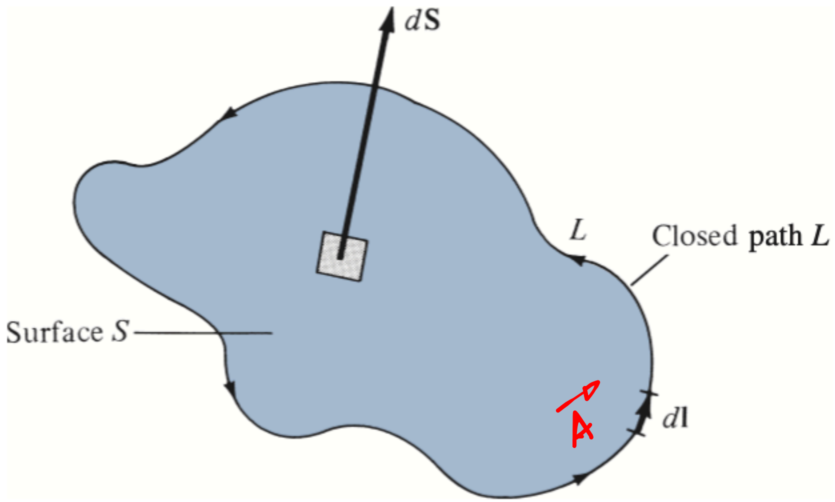


FIGURE 3.21 Determining the sense of $d\mathbf{l}$ and $d\mathbf{S}$ involved in Stokes's theorem.

$\int$ $\oiint$

Line → Surface

Also, from the definition of the curl of **A** in eq. (3.45), we may expect that

curl A

$$\boxed{\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}} \qquad (3.57)$$

Gauss's

Divergence Theorem

The **divergence theorem** states that the total outward flux of a vector field **A** through the *closed* surface *S* is the same as the volume integral of the divergence of **A**.

$$\boxed{\oint_S \mathbf{A}\cdot d\mathbf{S} = \int_v \nabla\cdot\mathbf{A}\,dv}$$

$\oiint$ $\iiint$

surface $\rightarrow$ Volume

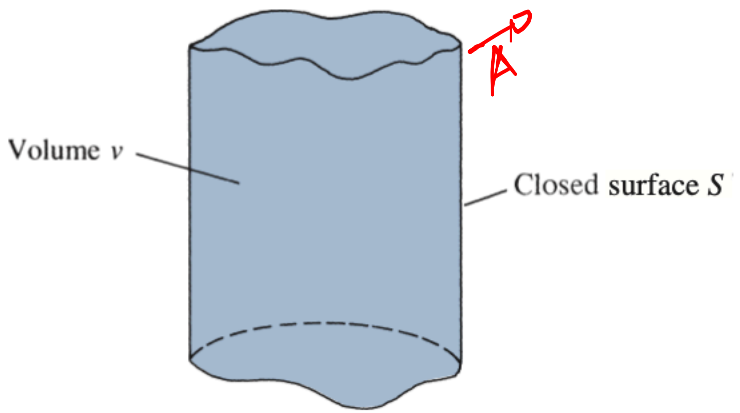


FIGURE 3.17 Volume *v* enclosed by surface *S*.

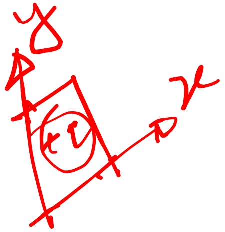
Maxwell's First Equation

$+q$ Ψ

sphere

Gauss's law states that the total electric flux through any closed surface is equal to the total charge enclosed by that surface.

Scalar electric flux (ψ)
They are the imaginary lines of force radiating in outward direction
A charge can be source or sink



$$\Psi = Q_{\text{enc}}$$

$\Psi = DA \;\Big|\; D = \dfrac{\Psi}{A}$

that is,

$$\Psi = \oint_S d\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

$$= \text{total charge enclosed } Q = \int_v \rho_v \, dv$$

charge density

$\dfrac{Q}{V}$

or

$\nabla \cdot D$

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = \boxed{\int_v \rho_v \, dv}$$

Gauss's

By applying divergence theorem to the middle term in eq. (4.41), we have

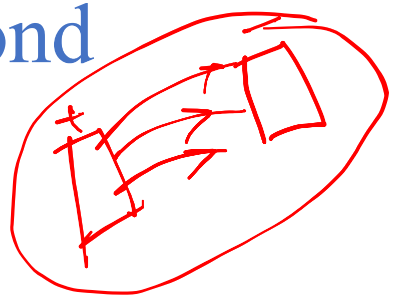
$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \boxed{\int_v \nabla \cdot \mathbf{D} \, dv}$$

Comparing the two volume integrals in eqs. (4.41) and (4.42) results in

$$\boxed{\rho_v = \nabla \cdot \mathbf{D}}$$

$\nabla \cdot B = ?$

Maxwell's Second Equation



Scalar magnetic flux (ϕ)
They are the circular magnetic field generated around a current carrying conductor.
No source/sink

$$\oiint \vec{B}.ds = \phi_{enclosed} \text{ --(1)}$$

$$\oiint \vec{B}.ds = 0 \text{ ---(2)}$$

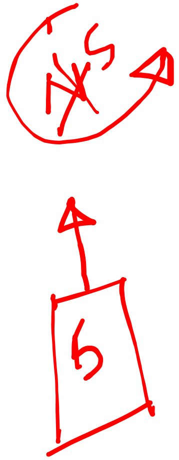
$$\Rightarrow \oiint \vec{B}.ds = \iiint \nabla.\vec{B}dv \text{ ---(3)}$$

$$\iiint \nabla.\vec{B}dv = 0 \text{ --(4)}$$

$$\nabla.\vec{B} = 0$$

Maxwell's Third Equation

Faraday discovered that the induced emf, V(emf in volts) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.



$$\boxed{V_{emf} = -\frac{d\Psi}{dt}}$$

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

Applying Stoke's Theorem,

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$