Lecture 1

Basics and Mathematical Formulations of Electrodynamics

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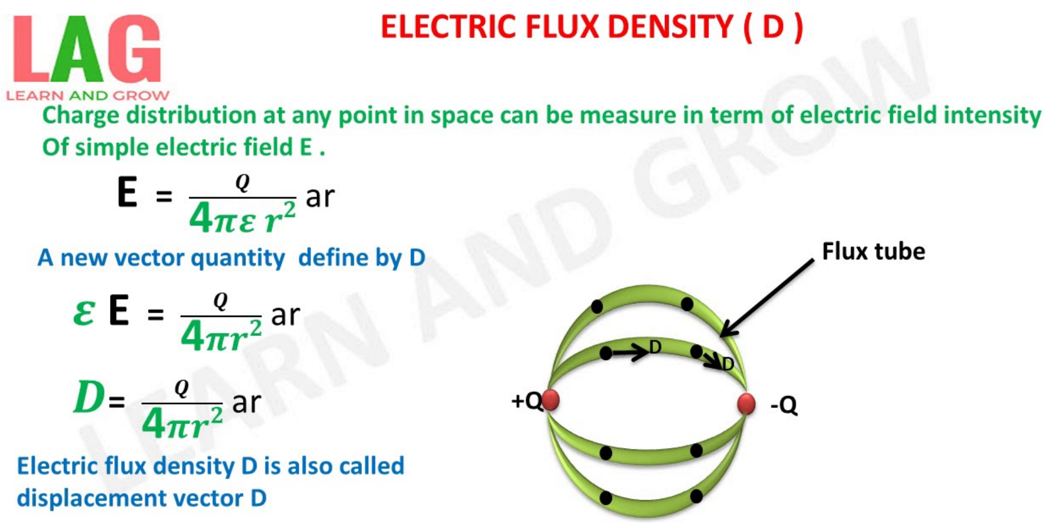
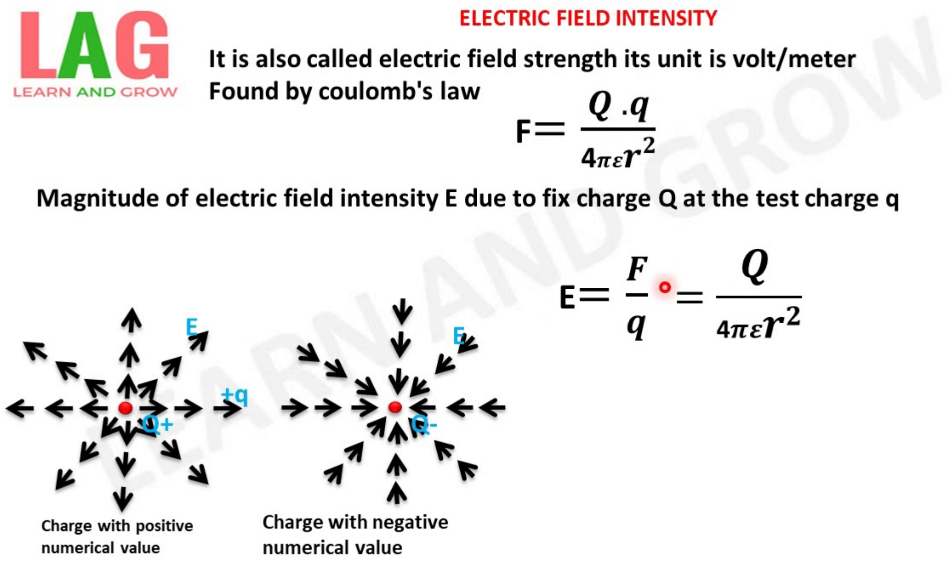
Electric Flux Density(D) and Electric Field Intensity(E)

Electric flux density is electric flux passing through a unit area perpendicular to the direction of the flux.

The electric field Intensity at a point is the force experienced by a unit positive charge placed at that point.

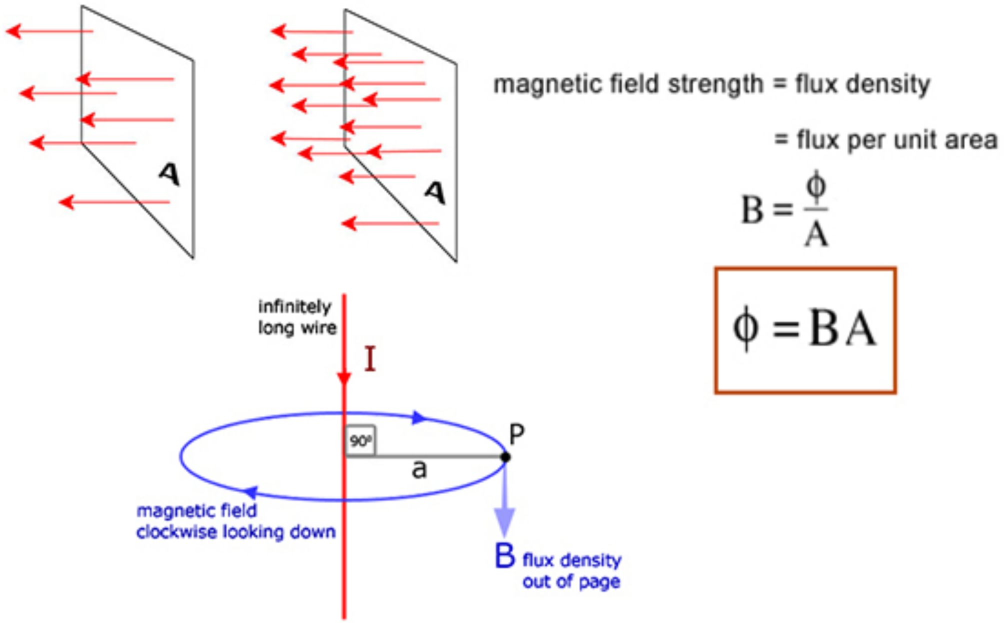


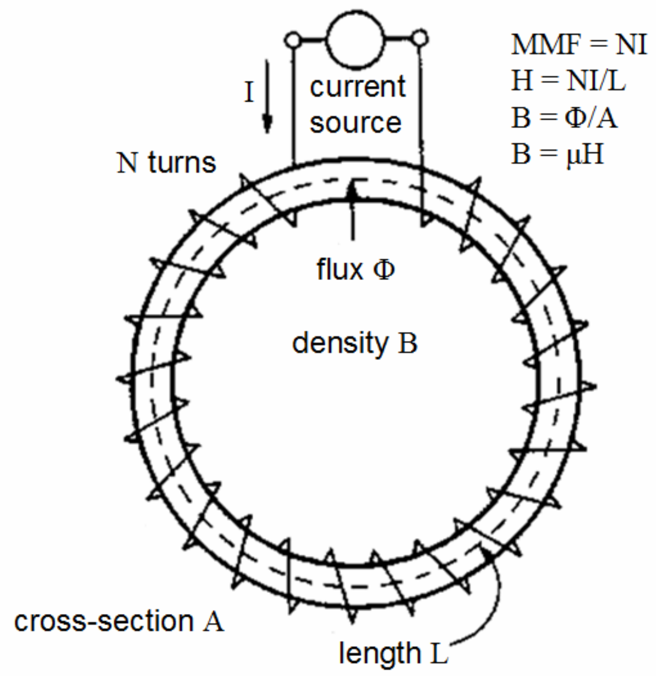


Magnetic Flux Density(B) and Magnetic Field Intensity(H)

Magnetic flux density(B) is defined as the amount of magnetic flux in an area taken perpendicular to the magnetic flux's direction.

The Magnetic field intensity(H) is a ratio of the MMF needed to create a certain Flux Density (B) within a particular material per unit length of that material.

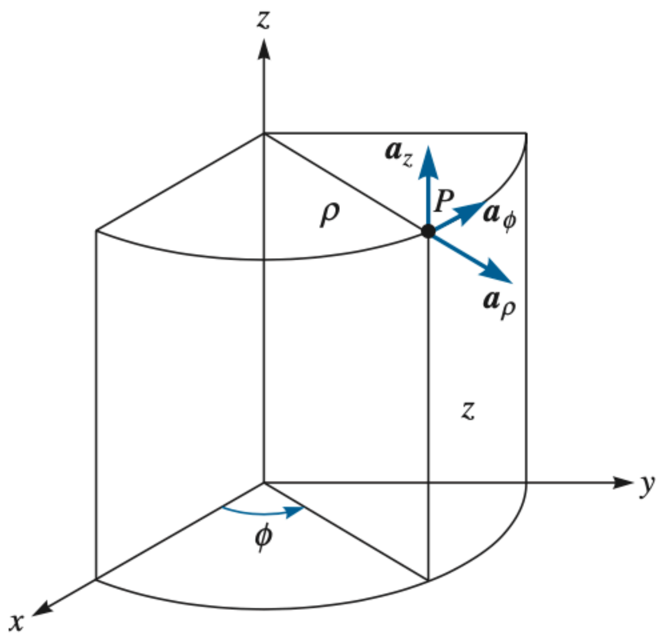




Circular Cylindrical Coordinates

$$\rho = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\frac{y}{x}, \quad z = z$$

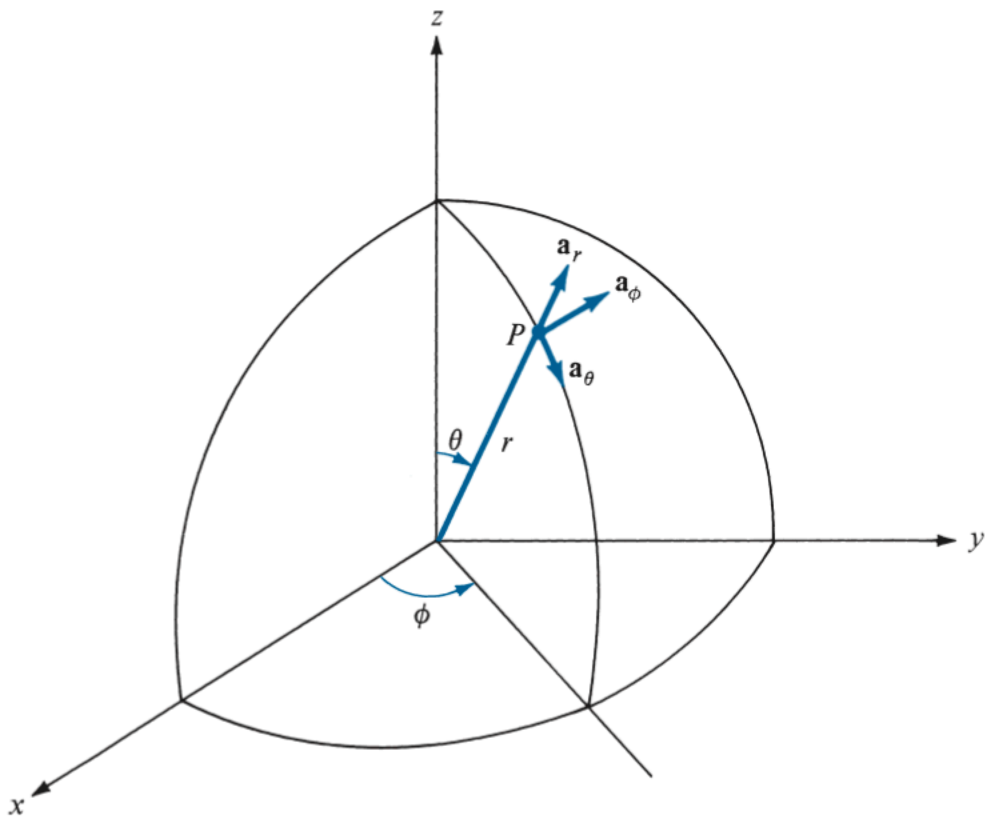
$$x = \rho \cos\phi, \quad y = \rho \sin\phi, \quad z = z$$



Spherical Coordinates

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \quad \phi = \tan^{-1} \frac{y}{x}$$

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta$$



Given the vector field

$$\mathbf{H} = \rho z \cos\phi\, \mathbf{a}_\rho + e^{-2} \sin\frac{\phi}{2}\, \mathbf{a}_\phi + \rho^2 \mathbf{a}_z$$

at point $(1, \pi/3, 0)$, find

(a) $\mathbf{H} \cdot \mathbf{a}_x$

(b) $\mathbf{H} \times \mathbf{a}_\theta$

(c) The vector component of $\mathbf{H}$ normal to surface $\rho = 1$

(d) The scalar component of $\mathbf{H}$ tangential to the plane $z = 0$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

(a) $\vec{H} \cdot \hat{a}_x$

$\hat{a}_x = \cos\phi\, \hat{a}_\rho - \sin\phi\, \hat{a}_\phi$

$= -e^{-2} \sin\frac{\phi}{2} \times \sin\phi\, \hat{a}_\rho$

$\phi = \pi/3$

$= -0.0586\, \hat{a}_\phi$

(b) $\hat{a}_z = \cos\theta\, \hat{a}_r - \sin\theta\, \hat{a}_\theta$

$\hat{a}_\theta = \cos\theta\, \hat{a}_\rho - \sin\theta\, \hat{a}_z$

$$\vec{H} \times \hat{a}_\theta = \begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi & \hat{a}_z \\ 0 & 0.677 & 1 \\ 0 & 0 & -1 \end{vmatrix}$$

$= (0, 0.0677, 1)$

$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{\rho}{z} = \tan^{-1}(\infty) = 90^\circ$

Given the vector field

$$\mathbf{H} = \rho z \cos\phi \, \mathbf{a}_\rho + e^{-2} \sin\frac{\phi}{2} \, \mathbf{a}_\phi + \rho^2 \mathbf{a}_z$$

at point $(1, \pi/3, 0)$, find

(a) $\mathbf{H} \cdot \mathbf{a}_x$

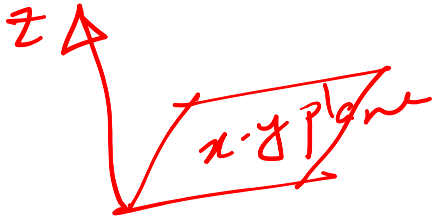
(b) $\mathbf{H} \times \mathbf{a}_\theta$

(c) The vector component of $\mathbf{H}$ normal to surface $\rho = 1$

(d) The scalar component of $\mathbf{H}$ tangential to the plane $z = 0$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$



(c) $(\vec{H} \cdot \hat{a}_\rho)\,\hat{a}_\rho$

$= 0$

(d) $\vec{H} \times \hat{a}_z$ ← always in xy plane

$$= \begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi & \hat{a}_z \\ 0 & 0.677 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

If $\mathbf{A} = 3\mathbf{a}_r + 2\mathbf{a}_\theta - 6\mathbf{a}_\phi$ and $\mathbf{B} = 4\mathbf{a}_r + 3\mathbf{a}_\phi$, determine

(a) $\mathbf{A} \cdot \mathbf{B}$ ✓

(b) $|\mathbf{A} \times \mathbf{B}|$ ✓

(c) The vector component of $\mathbf{A}$ along $\mathbf{a}_z$ at $(1, \pi/3, 5\pi/4)$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

(c) $\hat{a}_z = \cos\theta\, \hat{a}_\rho - \sin\theta\, \hat{a}_\theta$

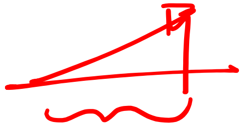
$= \cos(\pi/3)\, \hat{a}_\rho - \sin(\pi/3)\, \hat{a}_\theta$

$= 1/2\, \hat{a}_\rho - \sqrt{3}/2\, \hat{a}_\theta$

$(\vec{A} \cdot \hat{a}_z)\, \hat{a}_z = -0.232\left(1/2\, \hat{a}_\rho - \sqrt{3}/2\, \hat{a}_\theta\right)$

$= -0.116\, \hat{a}_\rho + 0.2009\, \hat{a}_\theta$

Component part

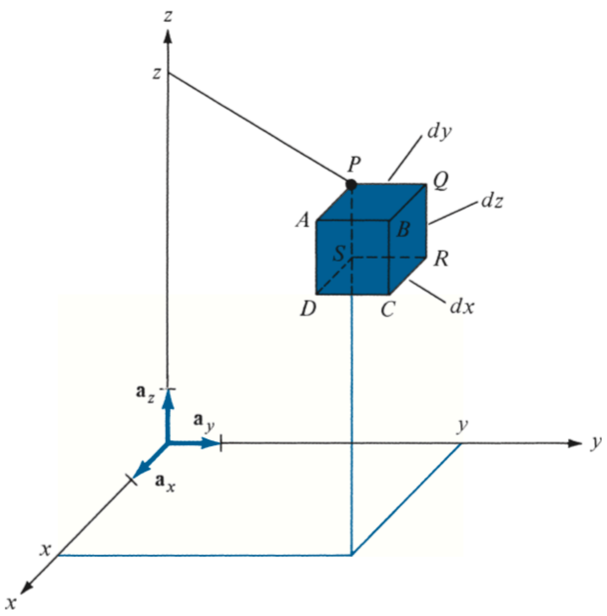


Component

Vector Calculus (Cartesian Co-ordinate System)

1. Differential displacement is given by

$$d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$$

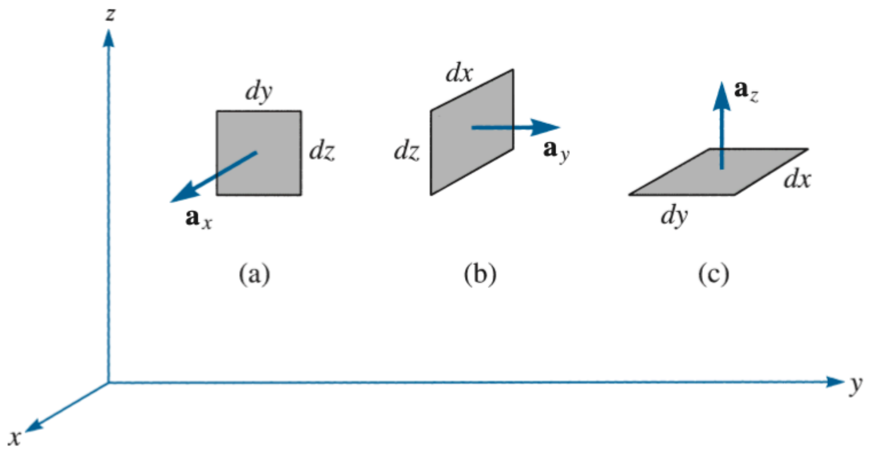


2. Differential normal surface area is given by

$$d\mathbf{S} = dy\,dz\,\mathbf{a}_x \\ dx\,dz\,\mathbf{a}_y \\ dx\,dy\,\mathbf{a}_z$$

3. Differential volume is given by

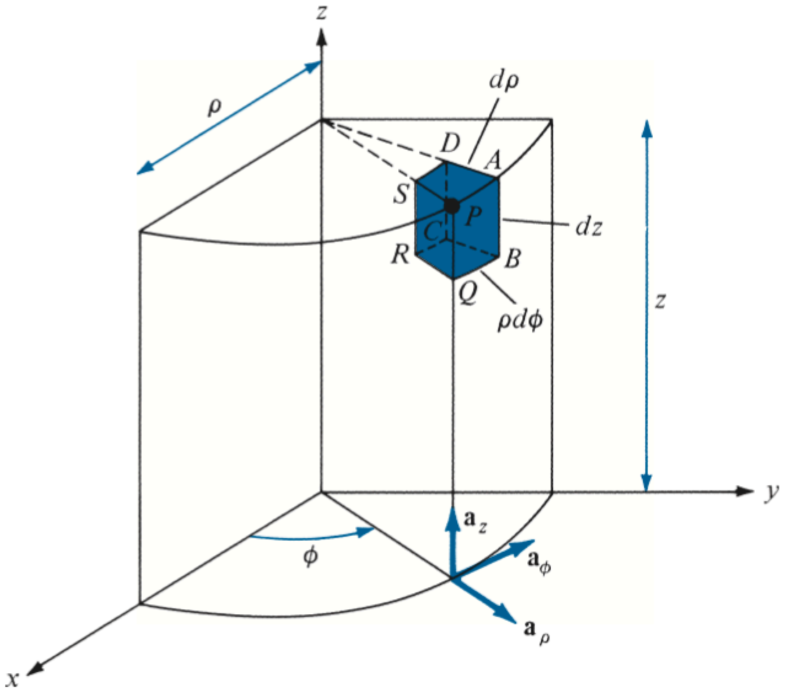
$$dv = dx\,dy\,dz$$



Vector Calculus (Cartesian Co-ordinate System)

1. Differential displacement is given by

$$d\mathbf{l} = d\rho\, \mathbf{a}_\rho + \rho\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z$$



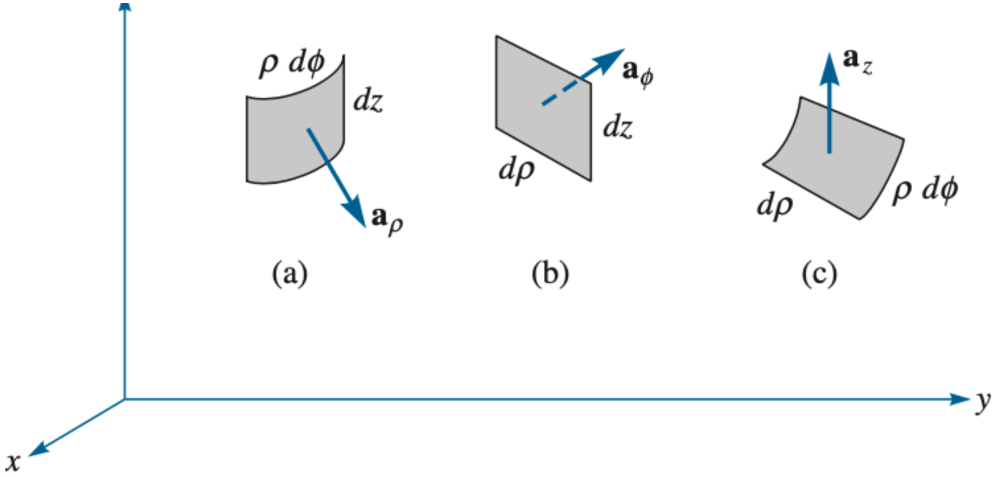
2. Differential normal surface area is given by

$$d\mathbf{S} = \begin{matrix} \rho\, d\phi\, dz\, \mathbf{a}_\rho \\ d\rho\, dz\, \mathbf{a}_\phi \\ \rho\, d\rho\, d\phi\, \mathbf{a}_z \end{matrix}$$

and illustrated in Figure 3.4.

3. Differential volume is given by

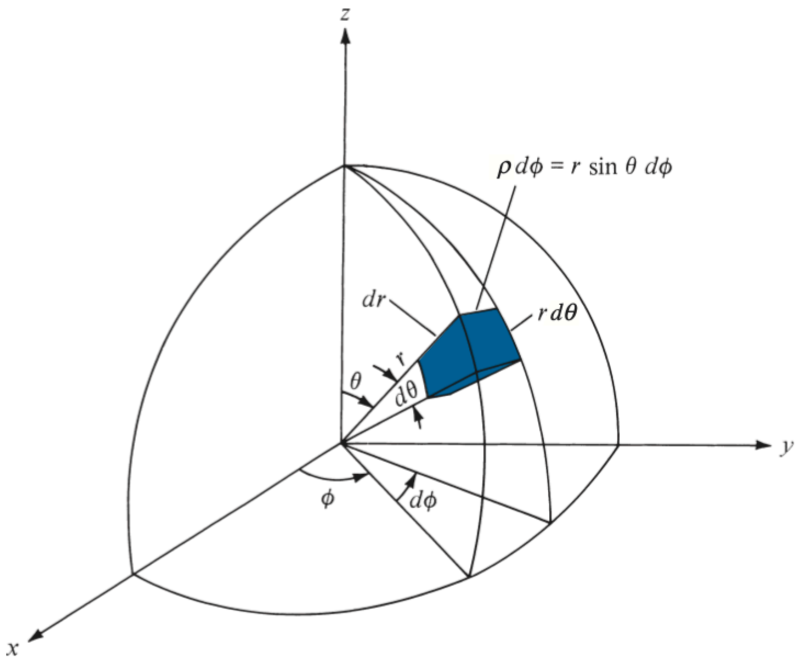
$$dv = \rho\, d\rho\, d\phi\, dz$$



Vector Calculus (Spherical Co-ordinate System)

1. The differential displacement is

$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$

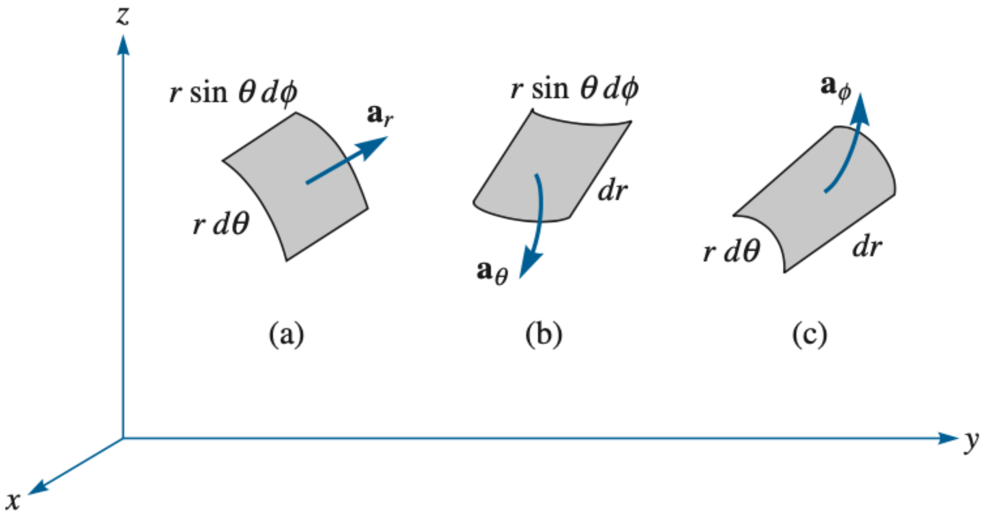


2. The differential normal surface area is

$$d\mathbf{S} = r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r$$
$$r \sin\theta\, dr\, d\phi\, \mathbf{a}_\theta$$
$$r\, dr\, d\theta\, \mathbf{a}_\phi$$

3. The differential volume is

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$



Calculate the circulation of

$$\mathbf{A} = \rho \cos\phi \, \mathbf{a}_\rho + z \sin\phi \, \mathbf{a}_z$$

around the edge L of the wedge defined by $0 \le \rho \le 2$, $0 \le \phi \le 60°$, $z = 0$ and shown in Figure 3.12.

$$\int_{①} F \cdot dl = \int_0^2 \rho \cos\phi \, \hat{a}_\rho \quad (\phi = 0°)$$

$$= \int_0^2 \rho \, \hat{a}_\rho$$

$$= 2$$

$$\int_{②} F \cdot dl = \int_{\phi=0°}^{60°} \rho \, d\phi \, \hat{a}_\phi$$

No component of $\hat{a}_\phi$ → so zero

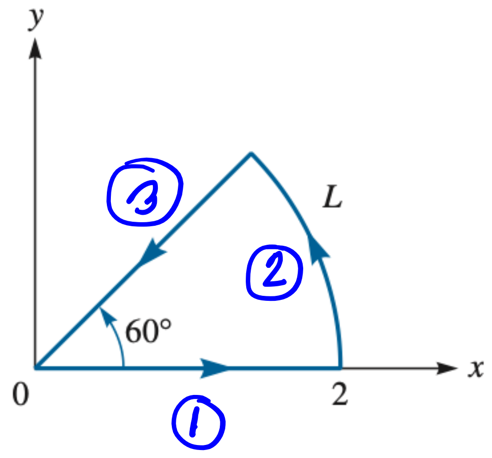
$$= 0$$

$$\int_{③} \vec{F} \cdot dl = \int_2^0 \rho \cos\phi \, \hat{a}_\rho \quad (\phi = 60°)$$

$$= -1$$

Total circulation $= 2 - 1$

$= 1$



The Del Operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

← Cartesian Co-ordinate System

$$\nabla = \mathbf{a}_\rho \frac{\partial}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \mathbf{a}_z \frac{\partial}{\partial z}$$

← Cylindrical

$$\nabla = \mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

← Spherical

Gradient

The Gradient of a scalar field V is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

be expressed in Cartesian, cylindrical, and spherical coordinates. For Cartesian coordinates

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

for cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \tag{3.29}$$

and for spherical coordinates,

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \tag{3.30}$$

Divergence

The divergence of vector A at a given point P is the outward flux per unit volume as the volume shrinks about P.

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Cylindrical →

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

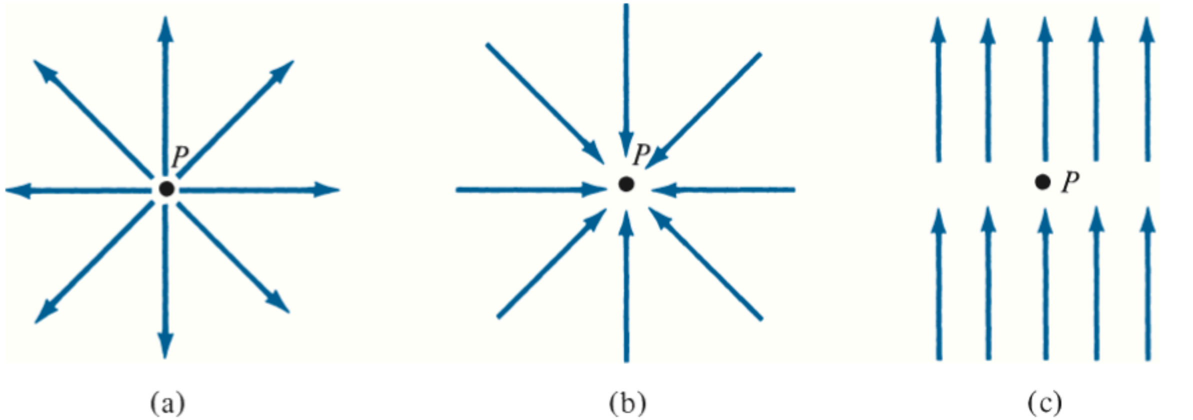


FIGURE 3.15 Illustration of the divergence of a vector field at *P*: (**a**) positive divergence, (**b**) negative divergence, (**c**) zero divergence.

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial \mathbf{A}_\phi}{\partial \phi}$$

→ Spherical (Formula bit different)

Curl

The Curl of vector A is an axial (or rotational) vector whose magnitude is the maximum circulation of A per unit area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

$$\text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right)_{\max} \mathbf{a}_n$$