Lecture 13

Antennas

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Antennas

- An antenna is a **device for transmitting or receiving** electromagnetic waves.
- An antenna transforms an electromagnetically guided wave (for example, guided by a coaxial line, a planar microstrip line, or a waveguide) into an electromagnetic wave in free space and vice versa. The first is a transmitting antenna and the latter is a receiving antenna.
- The antenna dimensions may range from a portion of a wavelength (e.g., a short dipole) up to several thousands of wavelengths (e.g., a radio telescope).
- Depending on the antenna geometry, the antenna may focus the radiated power into specific spatial directions; this is described by the antenna's directivity and gain.

Antennas

The most important characteristics of an antenna are:

- the three-dimensional antenna radiation characteristic
- the antenna gain over frequency
- the center frequency of operation
- the frequency bandwidth
- the mechanical dimensions (WxHxD)
- the polarization (linear, circular, elliptical polarization)
- the input impedance

Antennas

dimensions < λ

- short dipoles
- frame antennas
- loop antennas
- …

dimensions ~ λ
> λ
>> λ

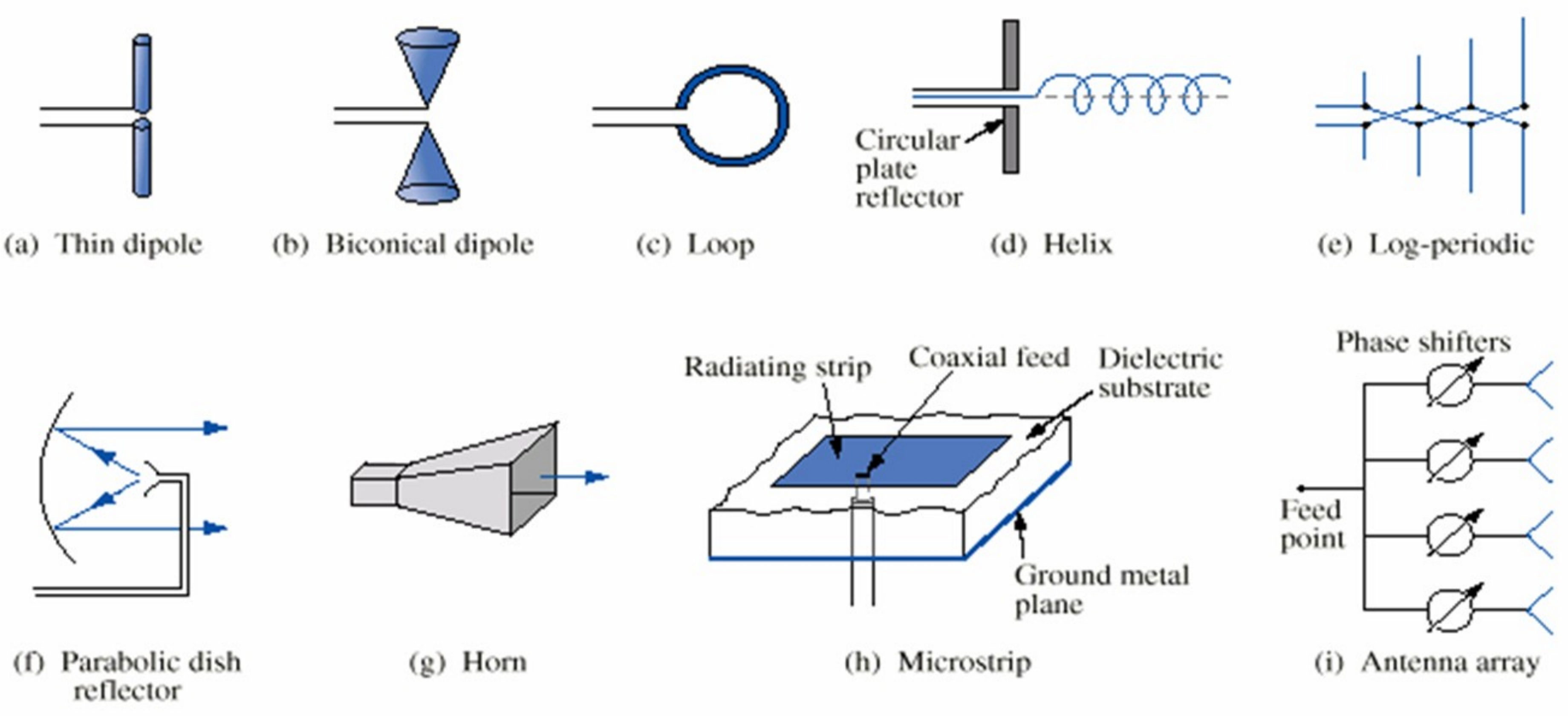
orientation $\perp$ to direction of radiation

- dipole arrays
- aperture antennas
    - reflector antennas (parabolic, Cassegrain, Gregory)
    - horn antennas
    - lens antennas
    - patch antennas, patch arrays

orientation || to direction of radiation

- dielectric rod antennas
- Yagi antennas
- …

Antennas



Circular plate reflector
(a) Thin dipole
(b) Biconical dipole
(c) Loop
(d) Helix
(e) Log-periodic
Radiating strip
Coaxial feed
Dielectric substrate
Ground metal plane
Phase shifters
Feed point
(f) Parabolic dish reflector
(g) Horn
(h) Microstrip
(i) Antenna array

Retarded Potentials $\vec{A}$, Φ

(also called Electrodynamic Potentials)

Remember, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{P}) = 0$, $\vec{E} = -\vec{\nabla} V$

Curl $\vec{A}$

Any Vector

$\vec{E} = -\vec{\nabla} V$

With Maxwell's fourth equation

$$\text{div}\,\vec{B} = 0 \tag{5.1}$$

we can describe the magnetic flux density $\vec{B}$ as a curl field of a vector potential $\vec{A}$,

$$\vec{B} = \text{curl}\,\vec{A} \tag{5.2}$$

$\vec{A}$ = Magnetic Vector Potential

because the divergence of a curl field always vanishes (a curl field cannot have sources), $\text{div}\,\text{curl}\,\vec{A} = 0$ (see eq. (2.39)).

Magnetic vector potential, A, is the vector quantity in classical electromagnetism defined so that its **curl is equal to the magnetic ~~field~~** flux density.

Retarded Potentials $\vec{A}$, Φ

(also called Electrodynamic Potentials)

Maxwell's second equation

$$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5.3}$$

can now be expressed in terms of $\vec{A}$:

$$\begin{aligned} \text{curl}\,\vec{E} &= -\frac{\partial(\text{curl}\,\vec{A})}{\partial t} \\ &= -\text{curl}\left(\frac{\partial \vec{A}}{\partial t}\right) \end{aligned} \tag{5.4}$$

or as

$$\text{curl}\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = \vec{0} \tag{5.5}$$

We also know from chapter 2, eq. (2.40), that the curl field of the gradient of a scalar field must always be zero, $\text{curl grad}\,\Phi = \vec{0}$ (a gradient field cannot have curls). With that, we introduce the scalar potential Φ with

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\text{grad}\,\Phi \tag{5.6}$$

- **Electric Scalar potential**, simply stated, describes **the situation where the difference in the potential** energies of an object in two different positions **depends only on the positions, not upon the path** taken by the object in traveling from one position to the other.

Let's Find the Differential Equation Using Retarded Potentials!

$$\vec{E} = -\text{grad}\,\Phi - \frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \text{curl}\,\vec{A}$$

Using Maxwell's third equation $\text{div}\,\vec{D} = \varrho$, we get a relation between $\vec{A}$ and Φ:

$$\begin{aligned}
\text{div}\,\vec{D} &= \varrho \\
\text{div}\,\vec{E} &= \frac{\varrho}{\varepsilon} \\
\text{div}\left(-\text{grad}\,\Phi - \frac{\partial \vec{A}}{\partial t}\right) &= \frac{\varrho}{\varepsilon} \\
\text{div}\,\text{grad}\,\Phi + \text{div}\left(\frac{\partial \vec{A}}{\partial t}\right) &= -\frac{\varrho}{\varepsilon}
\end{aligned}$$

or with the Laplace operator $\Delta = \text{div}\,\text{grad}$:

$$\boxed{\Delta\Phi + \text{div}\left(\frac{\partial \vec{A}}{\partial t}\right) = -\frac{\varrho}{\varepsilon}}$$

We also have to satisfy Maxwell's first equation

$$\begin{aligned}
\text{curl}\,\vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\
&= \vec{J} + \varepsilon\frac{\partial \vec{E}}{\partial t} \\
&= \vec{J} + \varepsilon\left(-\text{grad}\left(\frac{\partial \Phi}{\partial t}\right) - \frac{\partial^2 \vec{A}}{\partial t^2}\right)
\end{aligned}$$

Now, for the curls of $\vec{B}$ we get:

$$\begin{aligned}
\text{curl}\,\vec{B} &= \text{curl}\,(\mu\vec{H}) \\
&= \mu\vec{J} - \mu\varepsilon\,\text{grad}\left(\frac{\partial \Phi}{\partial t}\right) - \mu\varepsilon\frac{\partial^2 \vec{A}}{\partial t^2} \\
&= \text{curl}\,\text{curl}\,\vec{A}
\end{aligned}$$

and with the general relation (see eq. (2.38))

$$\text{curl}\,\text{curl}\,\vec{A} = \text{grad}\,\text{div}\,\vec{A} - \Delta\vec{A}$$

we finally obtain the differential equation

$$\boxed{-\Delta\vec{A} + \text{grad}\left(\text{div}\,\vec{A} + \mu\varepsilon\frac{\partial \Phi}{\partial t}\right) + \mu\varepsilon\frac{\partial^2 \vec{A}}{\partial t^2} = \mu\vec{J}}$$

Lorenz Gauge comes to Rescue
(To solve the differential equation!)

$$-\Delta\vec{A} + \text{grad}\left(\text{div}\,\vec{A} + \mu\varepsilon\frac{\partial\Phi}{\partial t}\right) + \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = \mu\vec{J}$$

To further simplify these equations, we consider the sources of $\vec{A}$ (i.e., div $\vec{A}$). Only if we know both the curls and the sources of a vector field is this vector field completely defined.
$\vec{B} = \text{curl}\,\vec{A}$ is not unambiguous, because also an

$$\tilde{\vec{A}} = \vec{A} + \text{grad}\,\Psi$$

yields the same $\vec{B}$ because

$$\text{curl}\,\text{grad}\,\Psi = \vec{0}$$

and

$$\text{curl}\,\tilde{\vec{A}} = \text{curl}\left(\vec{A} + \text{grad}\,\Psi\right) = \text{curl}\,\vec{A} + \text{curl}\,\text{grad}\,\Psi = \text{curl}\,\vec{A} = \vec{B}$$

This allows us to freely choose the sources of $\vec{A}$ and to do this in an advantageous manner. Therefore, we choose the sources of $\vec{A}$ as

$$\text{div}\,\vec{A} = -\mu\varepsilon\frac{\partial\Phi}{\partial t} \qquad \text{(Lorenz gauge)} \qquad (5.11)$$

Retarded Potentials (Two Decoupled Equation)

$$-\Delta\vec{A} + \text{grad}\left(\text{div}\,\vec{A} + \mu\varepsilon\frac{\partial\Phi}{\partial t}\right) + \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = \mu\vec{J}$$

$$\Delta\Phi + \text{div}\left(\frac{\partial\vec{A}}{\partial t}\right) = -\frac{\varrho}{\varepsilon}$$

$$\text{div}\,\vec{A} = -\mu\varepsilon\frac{\partial\Phi}{\partial t} \qquad \text{(Lorenz gauge)}$$

$$\Delta\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}$$

$$\Delta\Phi - \mu\varepsilon\frac{\partial^2\Phi}{\partial t^2} = -\frac{\varrho}{\varepsilon}$$

Retarded Potentials

Differential Equation → **And its solution**

$$\Delta\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J} \quad\Rightarrow\quad \vec{A}(P,t) = \frac{\mu}{4\pi}\iiint_V \frac{\vec{J}(P', t - \frac{R_{PP'}}{v})}{R_{PP'}}dV'$$

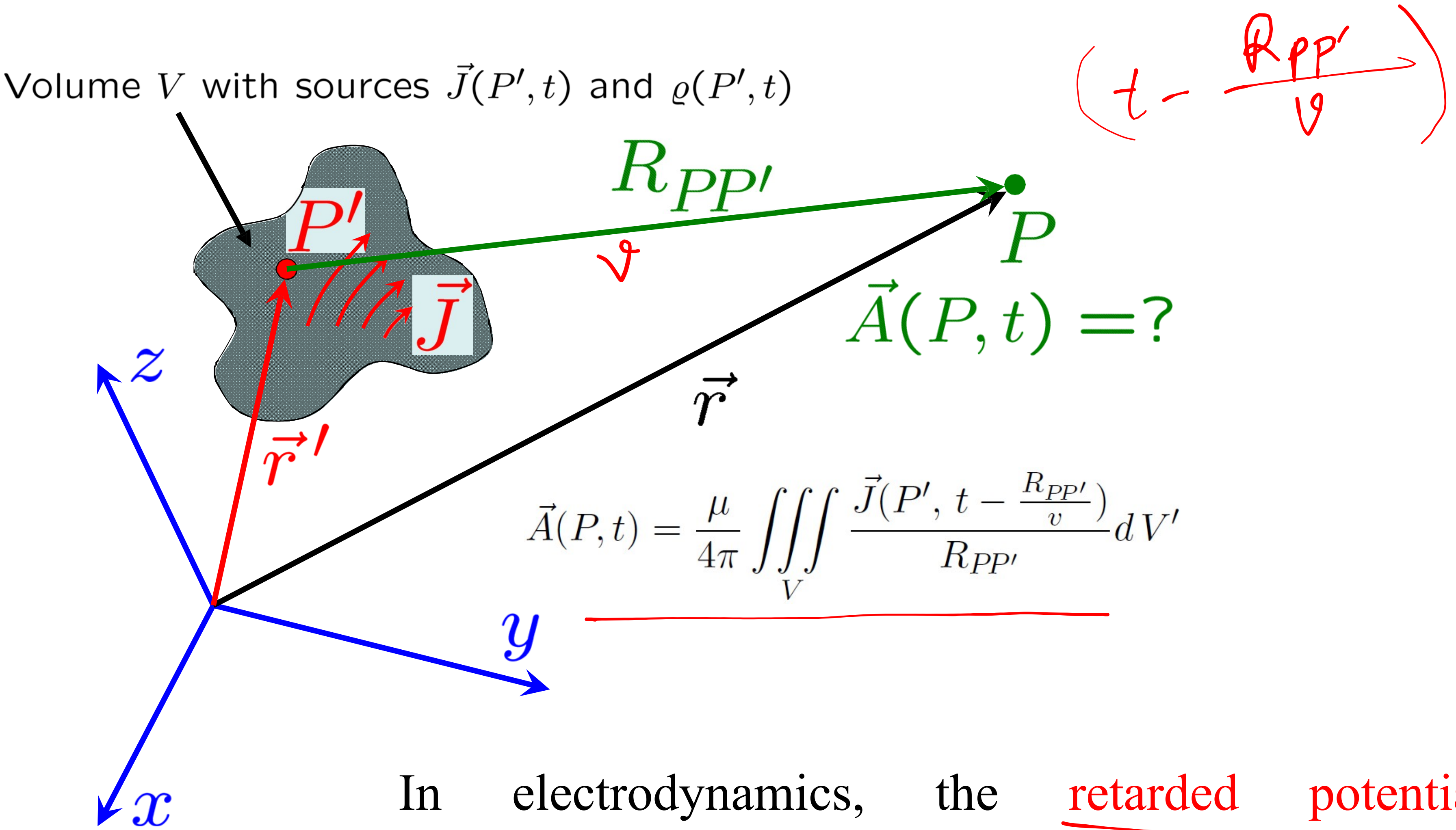
$$\Delta\Phi - \mu\varepsilon\frac{\partial^2\Phi}{\partial t^2} = -\frac{\varrho}{\varepsilon} \quad\Rightarrow\quad \Phi(P,t) = \frac{1}{4\pi\varepsilon}\iiint_V \frac{\varrho(P', t - \frac{R_{PP'}}{v})}{R_{PP'}}dV'$$

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\vec{E} = -\text{grad}\,\Phi - \frac{\partial\vec{A}}{\partial t}$$
$$\vec{B} = \text{curl}\,\vec{A}$$

Hurray! Now we can solve these equation!

Retarded Potentials



$$\vec{A}(P,t) = \frac{\mu}{4\pi} \iiint_V \frac{\vec{J}(P',\, t - \frac{R_{PP'}}{v})}{R_{PP'}} dV'$$

In electrodynamics, the retarded potentials are the electromagnetic potentials for the electromagnetic field generated by time-varying electric current or charge distributions in the past.

Retarded Potentials in Complex Notation

If the sources vary harmonically wrt. time, we prefer complex notation:

$$\underline{\vec{A}}(P) = \frac{\mu}{4\pi} \iiint\limits_V \frac{\underline{\vec{J}}(P') \cdot \mathrm{e}^{-jkR_{PP'}}}{R_{PP'}} dV'$$

$$\mathrm{e}^{j\omega(t-\frac{R}{v})} = \mathrm{e}^{j\omega t} \cdot \mathrm{e}^{-j\frac{\omega}{v}R}$$

(and with $\frac{\omega}{v} = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda} = k$ we get)

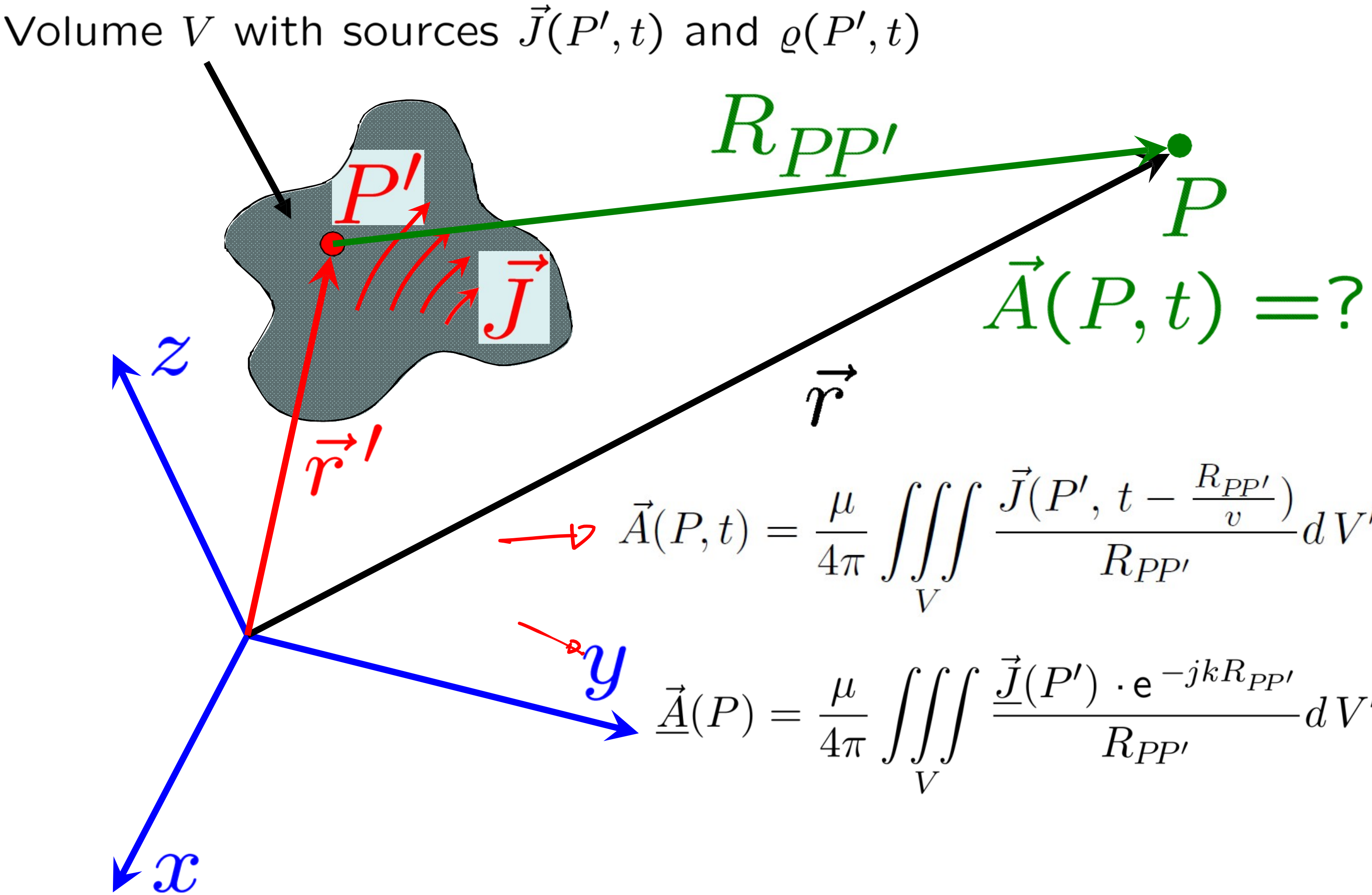
$$\mathrm{e}^{j\omega(t-\frac{R}{v})} = \mathrm{e}^{j\omega t} \cdot \mathrm{e}^{-jkR}$$

Retardation

$$\underline{\Phi}(P) = \frac{1}{4\pi\varepsilon} \iiint\limits_V \frac{\underline{\varrho}(P') \cdot \mathrm{e}^{-jkR_{PP'}}}{R_{PP'}} dV'$$

$$k = \omega\sqrt{\mu\varepsilon} \ \left(= \frac{\omega}{v}\right)$$

Retarded Potentials (Complex Notation)



$$\vec{A}(P,t) = \frac{\mu}{4\pi} \iiint_V \frac{\vec{J}\left(P', t - \frac{R_{PP'}}{v}\right)}{R_{PP'}} dV'$$

$$\underline{\vec{A}}(P) = \frac{\mu}{4\pi} \iiint_V \frac{\underline{\vec{J}}(P') \cdot \mathrm{e}^{-jkR_{PP'}}}{R_{PP'}} dV'$$

Final Take is :

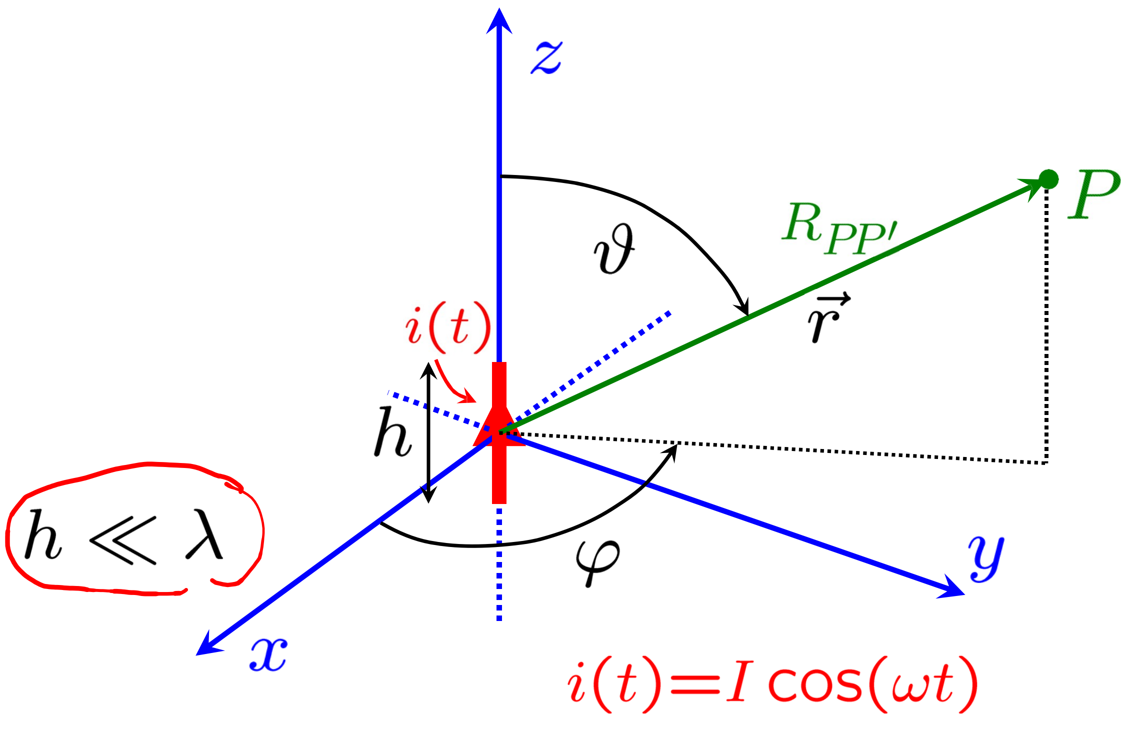
Then the potentials in complex notation read

$$\underline{\vec{A}}(P) = \frac{\mu}{4\pi} \iiint\limits_V \frac{\underline{\vec{J}}(P') \cdot \mathrm{e}^{-jkR_{PP'}}}{R_{PP'}} dV'$$

$$\underline{\Phi}(P) = \frac{1}{4\pi\varepsilon} \iiint\limits_V \frac{\underline{\varrho}(P') \cdot \mathrm{e}^{-jkR_{PP'}}}{R_{PP'}} dV'$$

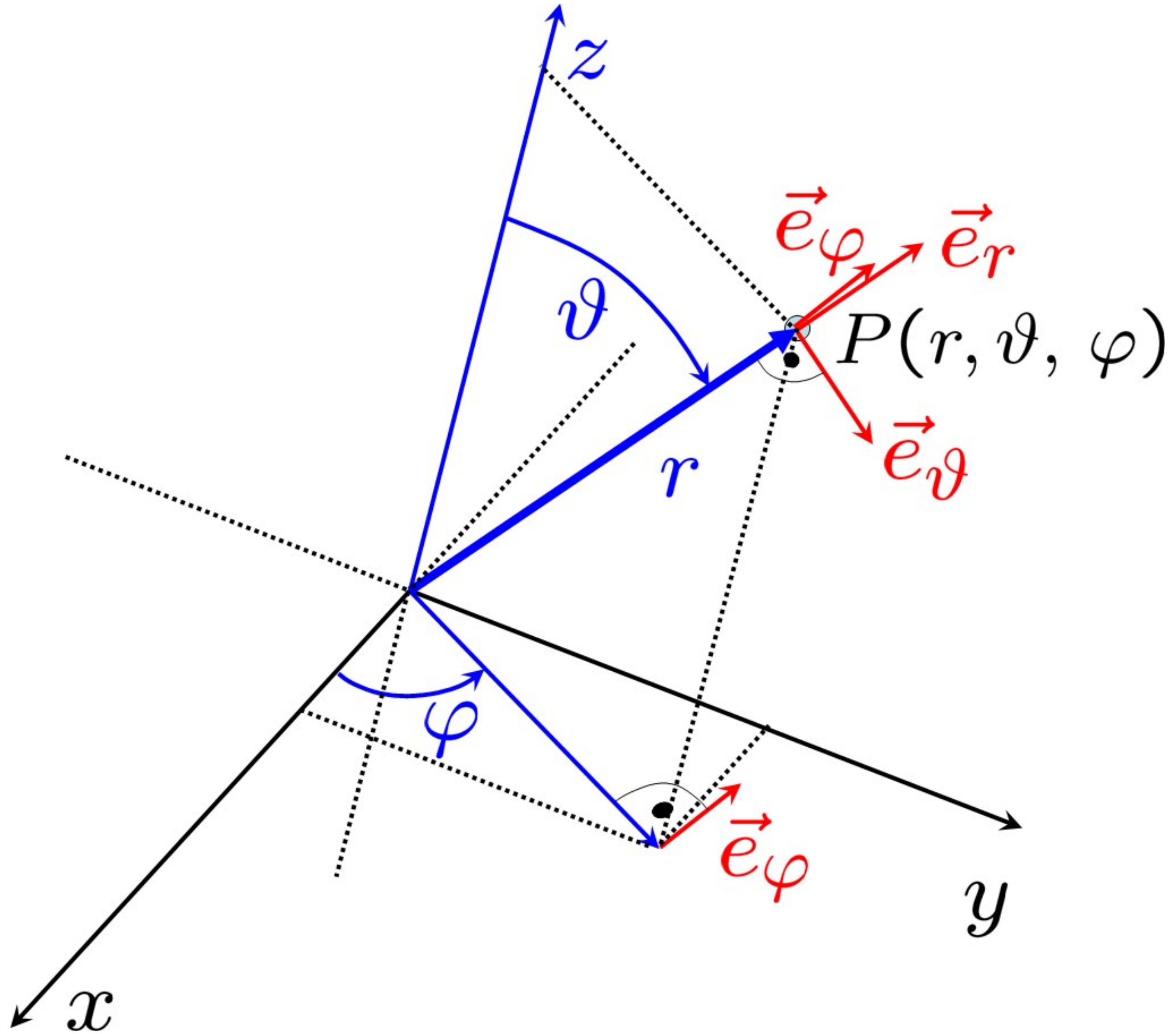
The Hertzian Dipole

r, θ, φ



❖ The **Hertzian dipole** is a theoretical dipole antenna that consists of an **infinitesimally small current** source acting in free-space. Although a true Hertzian dipole cannot physically exist, very short dipole antennas can make for a reasonable approximation.

2.6 Spherical Coordinate System



The Hertzian Dipole

Cartesian unity vectors in spherical coordinates:

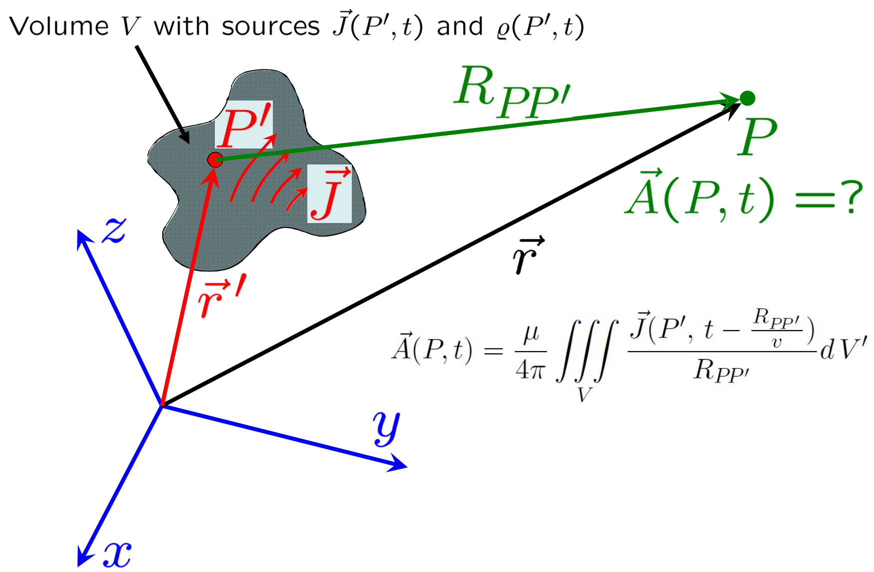
$$\begin{aligned} x &= r \sin\vartheta \cos\varphi \\ y &= r \sin\vartheta \sin\varphi \\ z &= r \cos\vartheta \end{aligned}$$

$$\vec{e}_x = ?, \quad \vec{e}_y = ?, \quad \vec{e}_z = ?$$

$$\text{grad}\,\Phi = \frac{\partial\Phi(r,\vartheta,\varphi)}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\Phi(r,\vartheta,\varphi)}{\partial\vartheta}\vec{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial\Phi(r,\vartheta,\varphi)}{\partial\varphi}\vec{e}_\varphi$$

$$\vec{e}_z = \text{grad}\, z = \cos\vartheta\,\vec{e}_r - \sin\vartheta\,\vec{e}_\vartheta$$

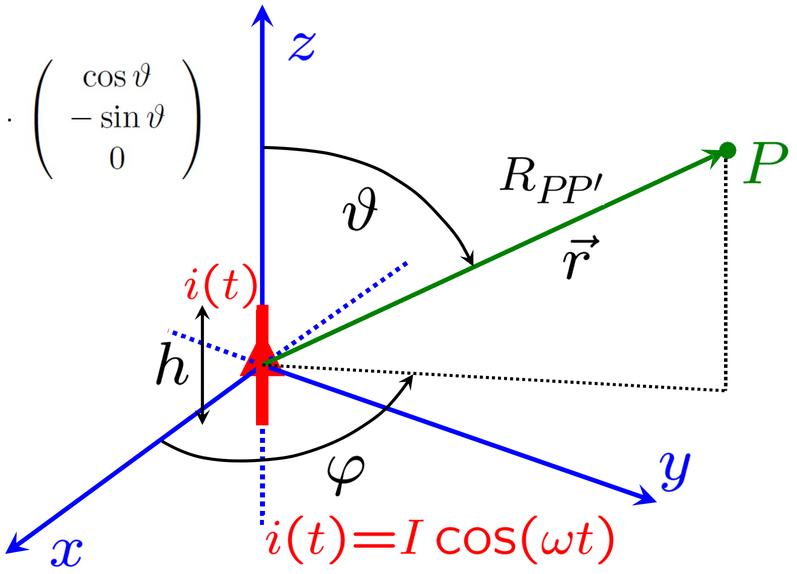
The Hertzian Dipole



$$\vec{A}(P,t) = \frac{\mu}{4\pi} \iiint\limits_V \frac{\vec{J}(P', t - \frac{R_{PP'}}{v})}{R_{PP'}} dV'$$

$$\underline{\vec{A}} = \frac{\mu\, I\, h}{4\pi} \cdot \frac{\mathrm{e}^{-jkr}}{r} \cdot \begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \\ 0 \end{pmatrix}$$

$h \ll \lambda$



$$\underline{\vec{A}}(P) = \frac{\mu}{4\pi} \iiint\limits_V \frac{\underline{\vec{J}}(P') \cdot \mathrm{e}^{-jkR_{PP'}}}{R_{PP'}} dV'$$

The dipole is oriented in the z-direction with a small height h ($h \ll \lambda$) and a current magnitude I. The current varies harmonically which allows us to use complex notation. The current density multiplied by an infinitesimal volume element reads

$$\underline{\vec{J}}\, dV = \underline{I}\, dz\, \vec{e}_z \tag{5.21}$$

$$\underline{\vec{A}} = \underline{A}_z\, \vec{e}_z \tag{5.22}$$

Because the dipole is placed at the origin and it is very small, the distance between the source point P' and an arbitrary point P is

$$R_{PP'} \approx r \tag{5.23}$$

$$\begin{aligned} \underline{\vec{A}}(P) &= \frac{\mu}{4\pi} \int_{-h/2}^{h/2} \frac{\underline{I}\, \mathrm{e}^{-jkr}}{r}\, dz\, \vec{e}_z \\ &= \frac{\mu\, \underline{I}\, h}{4\pi} \cdot \frac{\mathrm{e}^{-jkr}}{r}\, \vec{e}_z \end{aligned} \tag{5.24}$$

To transform $\underline{\vec{A}}$ from Cartesian coordinates into spherical coordinates we replace the unit vector $\vec{e}_z$ by

$$\vec{e}_z = \cos\vartheta\, \vec{e}_r - \sin\vartheta\, \vec{e}_\vartheta \tag{5.25}$$

and get $\underline{\vec{A}}$ in spherical coordinates as

$$\underline{\vec{A}} = \frac{\mu\, \underline{I}\, h}{4\pi} \cdot \frac{\mathrm{e}^{-jkr}}{r} \cdot \begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \\ 0 \end{pmatrix} \tag{5.26}$$

$A_\phi = 0$

The Hertzian Dipole

$$\underline{\vec{A}} = \frac{\mu\, I\, h}{4\pi} \cdot \frac{\mathrm{e}^{-jkr}}{r} \cdot \begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \\ 0 \end{pmatrix}$$

With

$$\underline{\vec{H}} = \frac{1}{\mu} \operatorname{curl} \underline{\vec{A}}$$

and

$$\operatorname{curl} \underline{\vec{A}} = \begin{pmatrix} \frac{1}{r\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}(\sin\vartheta\, \underline{A}_\varphi) - \frac{\partial \underline{A}_\vartheta}{\partial\varphi}\right) \\ \frac{1}{r\,\sin\vartheta}\frac{\partial \underline{A}_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r\, \underline{A}_\varphi) \\ \frac{1}{r}\frac{\partial}{\partial r}(r\, \underline{A}_\vartheta) - \frac{1}{r}\frac{\partial \underline{A}_r}{\partial\vartheta} \end{pmatrix}$$

we can see that the curl $\underline{\vec{A}}$ and therefore the magnetic field $\underline{\vec{H}}$ can only have a φ-component,

$$\underline{\vec{H}} = \underline{H}_\varphi \cdot \vec{e}_\varphi \tag{5.27}$$

$$\underline{H}_\varphi = \frac{I\, h}{4\pi} \sin\vartheta \cdot \mathrm{e}^{-jkr} \cdot \left(\frac{jk}{r} + \frac{1}{r^2}\right) \tag{5.28}$$

The Hertzian Dipole

$$\underline{H}_\varphi = \frac{I\,h}{4\pi}\,\sin\vartheta \cdot \mathrm{e}^{-jkr} \cdot \left(\frac{jk}{r} + \frac{1}{r^2}\right)$$

$$\vec{\underline{E}} = \frac{1}{j\omega\varepsilon}\,\mathrm{curl}\,\vec{\underline{H}}$$

$$\mathrm{curl}\,\vec{\underline{H}} = \begin{pmatrix} \frac{1}{r\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\underline{H}_\varphi) - \frac{\partial \underline{H}_\vartheta}{\partial\varphi}\right) \\ \frac{1}{r\,\sin\vartheta}\frac{\partial \underline{H}_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r\,\underline{H}_\varphi) \\ \frac{1}{r}\frac{\partial}{\partial r}(r\,\underline{H}_\vartheta) - \frac{1}{r}\frac{\partial \underline{H}_r}{\partial\vartheta} \end{pmatrix}$$

$$\underline{E}_r = \frac{I\,h}{2\pi}\cos\vartheta\,\mathrm{e}^{-jkr}\left(\frac{\sqrt{\mu/\varepsilon}}{r^2} + \frac{1}{j\omega\varepsilon\,r^3}\right)$$

$$\underline{E}_\vartheta = \frac{I\,h}{4\pi}\sin\vartheta\,\mathrm{e}^{-jkr}\left(\frac{j\omega\,\mu}{r} + \frac{\sqrt{\mu/\varepsilon}}{r^2} + \frac{1}{j\omega\varepsilon\,r^3}\right)$$

$$\underline{E}_\varphi = 0$$

The Hertzian Dipole

$$\underline{E}_r = \frac{I\,h}{2\pi}\cos\vartheta\, \mathrm{e}^{-jkr}\left(\frac{\sqrt{\mu/\varepsilon}}{r^2} + \frac{1}{j\omega\varepsilon\, r^3}\right)$$

$$\underline{E}_\vartheta = \frac{I\,h}{4\pi}\sin\vartheta\, \mathrm{e}^{-jkr}\left(\frac{j\omega\,\mu}{r} + \frac{\sqrt{\mu/\varepsilon}}{r^2} + \frac{1}{j\omega\varepsilon\, r^3}\right)$$

$$\underline{E}_\varphi = 0$$

$$\underline{E}_r = \frac{I\,h\,Z_F}{2\pi}\cos\vartheta\, \mathrm{e}^{-jkr}\left(\frac{1}{r^2} + \frac{1}{jk\,r^3}\right)$$

$$\underline{E}_\vartheta = \frac{I\,h\,Z_F}{4\pi}\sin\vartheta\, \mathrm{e}^{-jkr}\left(\frac{jk}{r} + \frac{1}{r^2} + \frac{1}{jk\,r^3}\right)$$

$$\underline{E}_\varphi = 0$$

$$k = \omega\sqrt{\mu\varepsilon} \qquad Z_F = \sqrt{\mu/\varepsilon}$$

The Hertzian Dipole (*Near Field Components*)

$$\frac{k}{r} \ll \frac{1}{r^2} \ll \frac{1}{k\,r^3}$$

$$\underline{H}_\varphi = \frac{I\,h}{4\pi} \sin\vartheta \cdot \mathrm{e}^{-jkr} \cdot \left(\frac{jk}{r} + \frac{1}{r^2}\right)$$

$$\underline{E}_r = \frac{I\,h\,Z_F}{2\pi} \cos\vartheta\, \mathrm{e}^{-jkr} \left(\frac{1}{r^2} + \frac{1}{jk\,r^3}\right)$$

$$\underline{E}_\vartheta = \frac{I\,h\,Z_F}{4\pi} \sin\vartheta\, \mathrm{e}^{-jkr} \left(\frac{jk}{r} + \frac{1}{r^2} + \frac{1}{jk\,r^3}\right)$$

$$\underline{E}_\varphi = 0$$

$$\underline{H}_r = 0$$

$$\underline{H}_\vartheta = 0$$

$$\underline{H}_\varphi = \frac{I\,h}{4\pi} \cdot \sin\vartheta \cdot \frac{1}{r^2}$$

$$\underline{E}_r = -j\frac{I\,h\,Z_F}{2\pi k} \cdot \cos\vartheta \cdot \frac{1}{r^3}$$

$$\underline{E}_\vartheta = -j\frac{I\,h\,Z_F}{4\pi k} \cdot \sin\vartheta \cdot \frac{1}{r^3}$$

$$\underline{E}_\varphi = 0$$

$r < 1$

$\frac{1}{0.1} < \frac{1}{0.1^2} < \frac{1}{0.1^3}$

$e^{-jkr} = e^{-0} = 1$

Out of phase

The Hertzian Dipole (*Far Field Components*)

$$\frac{k}{r} \gg \frac{1}{r^2} \gg \frac{1}{k\,r^3}$$

$$\underline{H}_\varphi = \frac{I\,h}{4\pi} \sin\vartheta \cdot \mathrm{e}^{-jkr} \cdot \left(\frac{jk}{r} + \frac{1}{r^2} \right)$$

$$\underline{E}_r = \frac{I\,h\,Z_F}{2\pi} \cos\vartheta \; \mathrm{e}^{-jkr} \left(\frac{1}{r^2} + \frac{1}{jk\,r^3} \right)$$

$$\underline{E}_\vartheta = \frac{I\,h\,Z_F}{4\pi} \sin\vartheta \; \mathrm{e}^{-jkr} \left(\frac{jk}{r} + \frac{1}{r^2} + \frac{1}{jk\,r^3} \right)$$

$$\underline{E}_\varphi = 0$$

$$\underline{H}_r = 0$$

$$\underline{H}_\vartheta = 0$$

$$\underline{H}_\varphi = j\frac{k\,I\,h}{4\pi} \cdot \frac{1}{r} \cdot \sin\vartheta \cdot \mathrm{e}^{-jkr}$$

$$\underline{E}_r = 0$$

$$\underline{E}_\vartheta = j\frac{k\,I\,h}{4\pi} \cdot Z_F \cdot \frac{1}{r} \cdot \sin\vartheta \cdot \mathrm{e}^{-jkr}$$

$$\underline{E}_\varphi = 0$$

1/10000 + 1/1

in phase!

The Hertzian Dipole

Far Field: a spherical TEM Wave

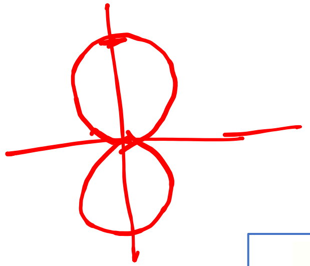
$$\vec{E} \perp \vec{H}$$

$$\frac{|\vec{E}|}{|\vec{H}|} = \frac{|E_\vartheta|}{|H_\varphi|} = \sqrt{\frac{\mu}{\varepsilon}} = Z_F = 377\,\Omega \quad \text{in free space}$$

$$\begin{aligned} \overline{\vec{S}} &= \frac{1}{2}\Re\left\{\underline{\vec{E}} \times \underline{\vec{H}}^*\right\} \\ &= \frac{1}{2}\Re\left\{\underline{E}_\vartheta\, \underline{H}_\varphi^*\right\}\, \vec{e}_r \end{aligned}$$

$$\overline{\vec{S}} = \frac{1}{2} Z_F \cdot \left(\frac{k\, I\, h}{4\pi}\right)^2 \cdot \frac{\sin^2\vartheta}{r^2}\, \vec{e}_r$$

Antenna Pattern of the Hertzian Dipole



$$f(\theta) = |\sin\theta|$$

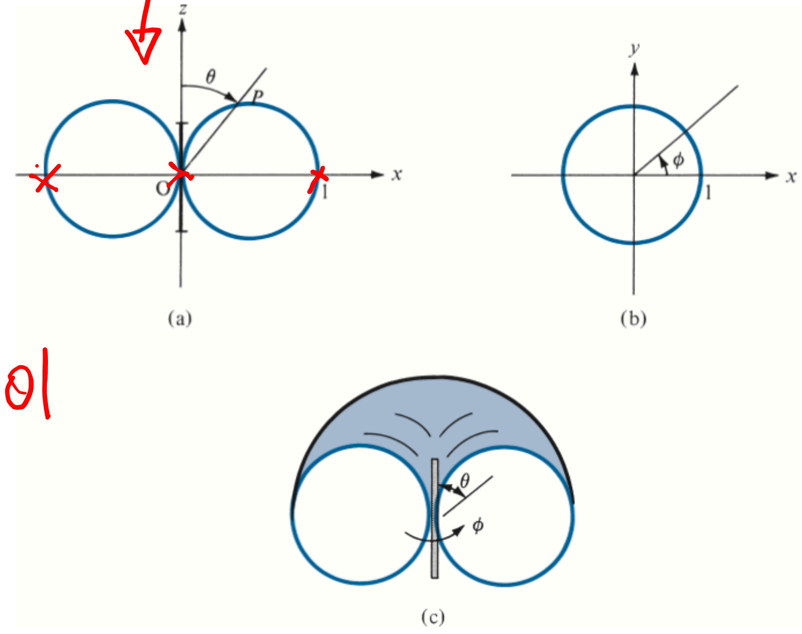


FIGURE 13.7 Field patterns of the Hertzian dipole: (**a**) normalized *E*-plane or vertical pattern (ϕ = constant = 0), (**b**) normalized *H*-plane or horizontal pattern ($\theta = \pi/2$), (**c**) three-dimensional pattern.

$$f^2(\theta) = \sin^2\theta$$

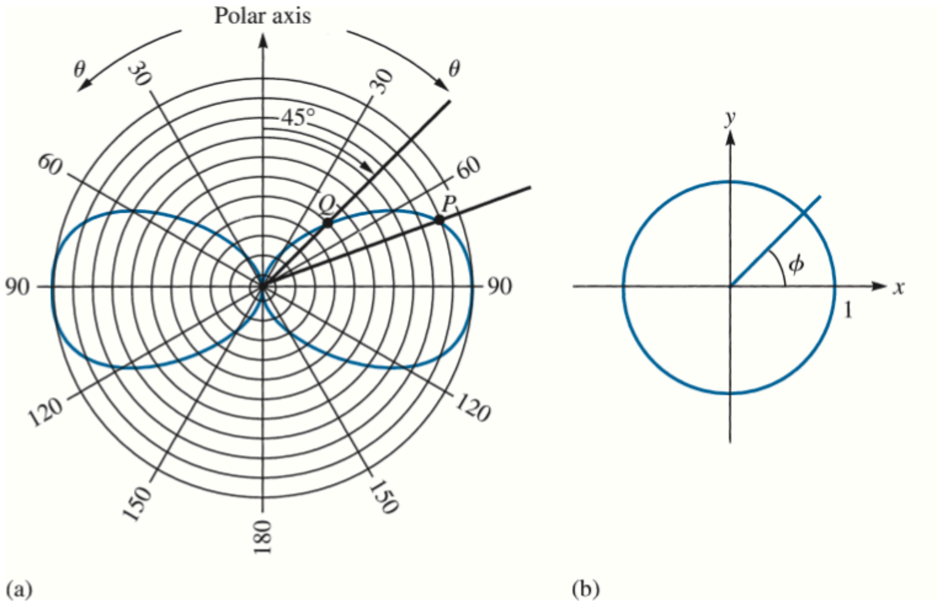
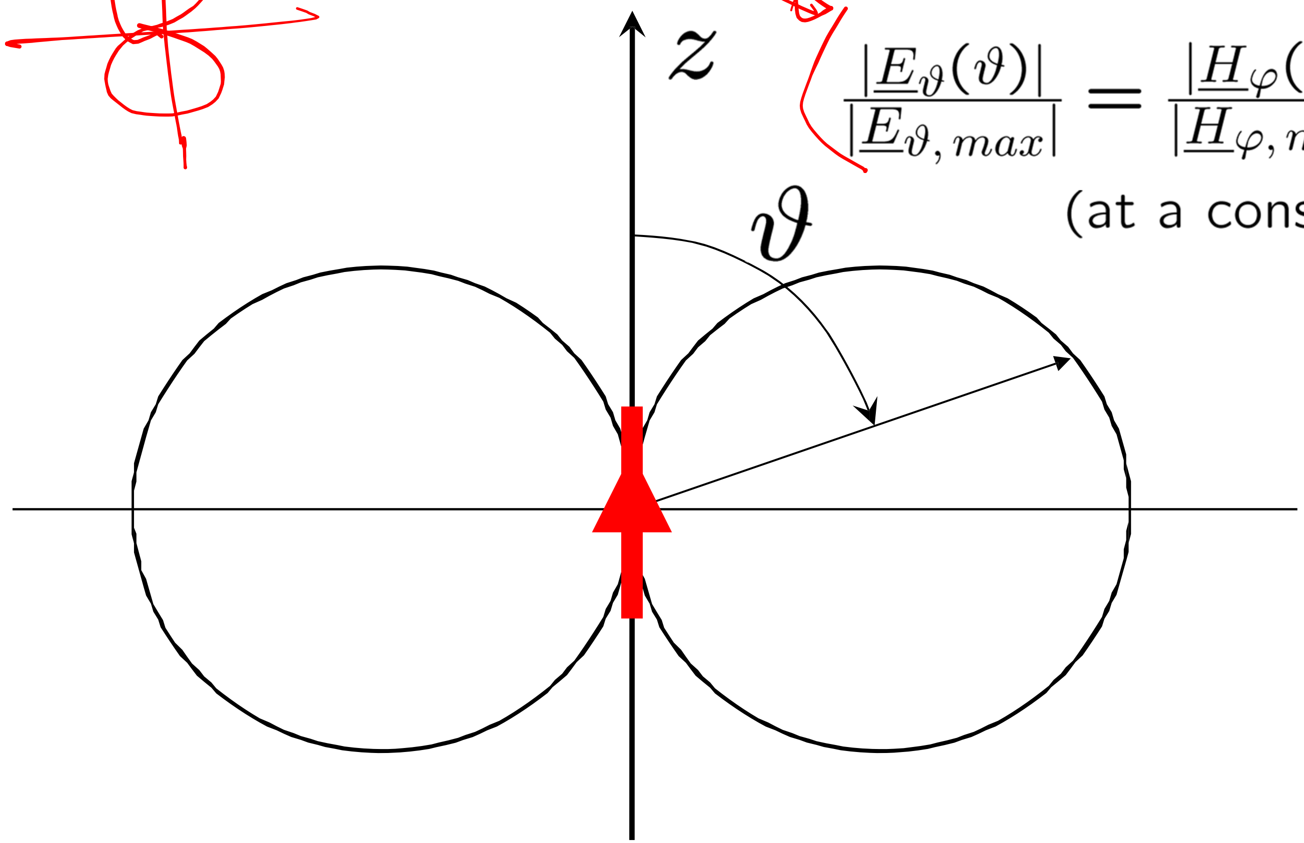


FIGURE 13.8 Power patterns of the Hertzian dipole: (**a**) (ϕ = constant = 0), (**b**) θ = constant = $\pi/2$.

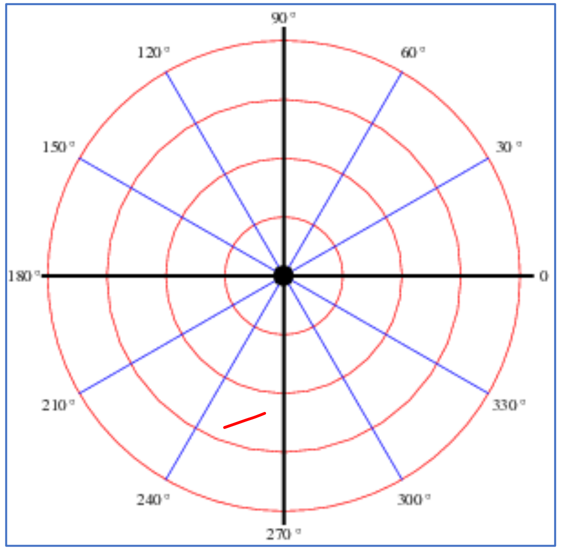
An **antenna pattern (or radiation patten)** is a **three dimensional plot** of its radiation at far field. When the **amplitude** of a specified component of the **E** field is plotted, it is called **the *field pattern* or *voltage pattern*.** When the **square of the amplitude** of **E** is plotted, it is called the ***power pattern***.

Radiation Characteristic of the Hertzian Dipole



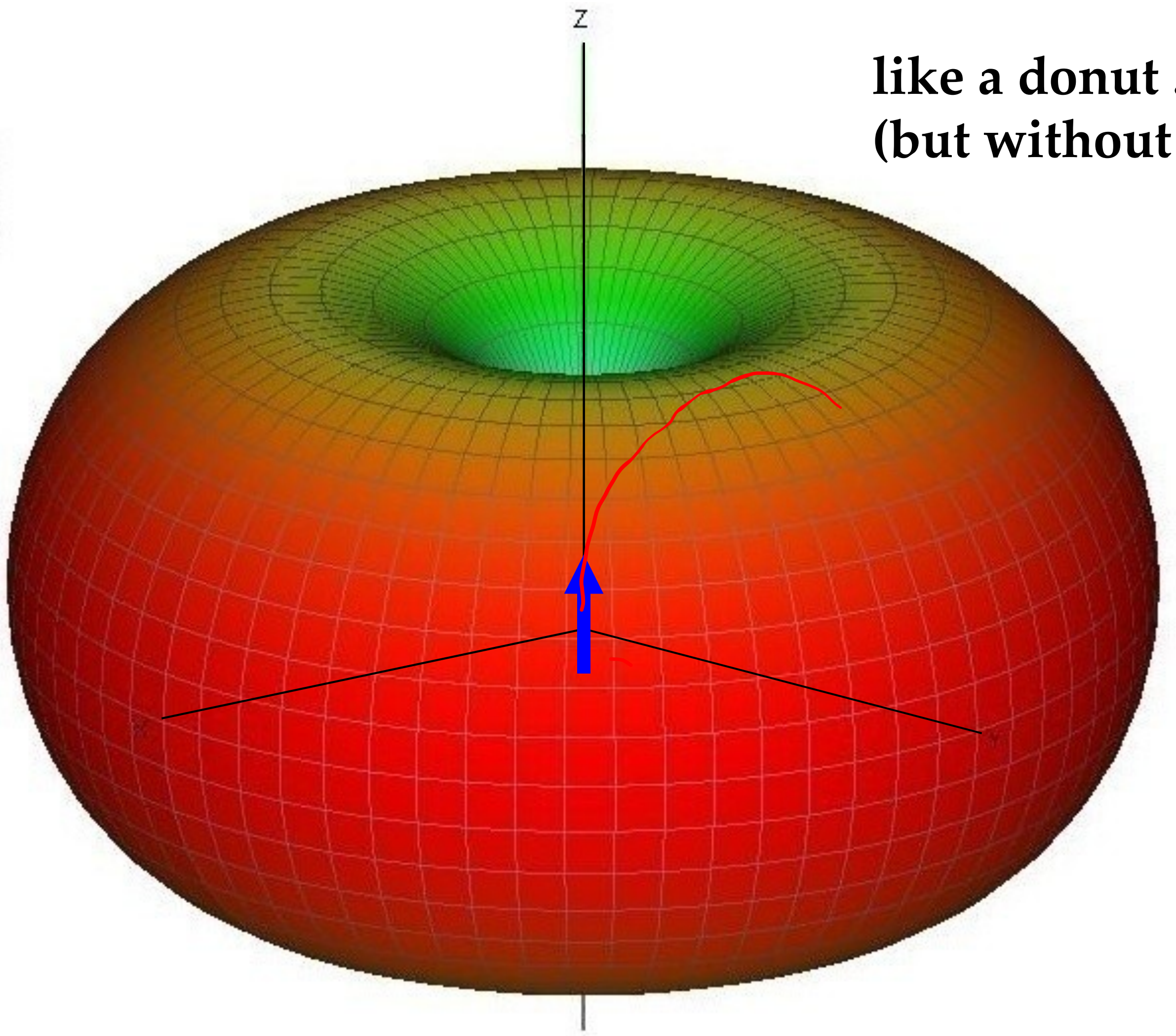
$$\frac{|E_\vartheta(\vartheta)|}{|E_{\vartheta,max}|} = \frac{|H_\varphi(\vartheta)|}{|H_{\varphi,max}|} = \sin\vartheta$$

(at a constant radius r)



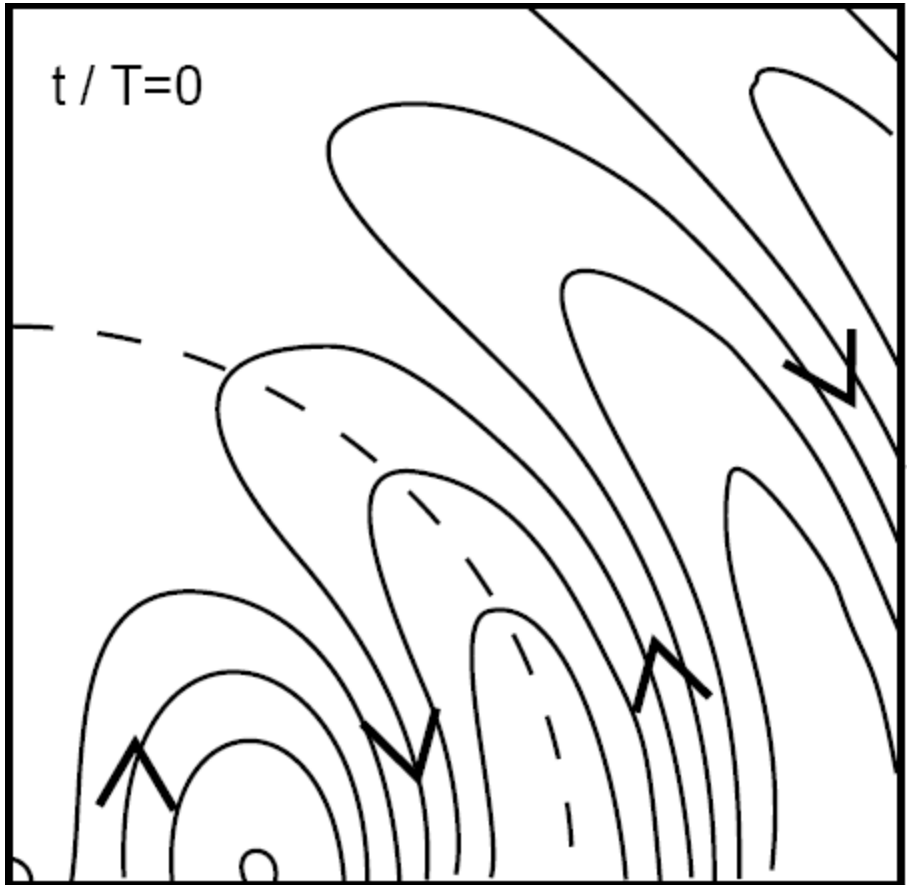
An antenna pattern (or radiation patten) is a **three dimensional plot** of its radiation at far field.

Radiation Characteristic of the Hertzian Dipole

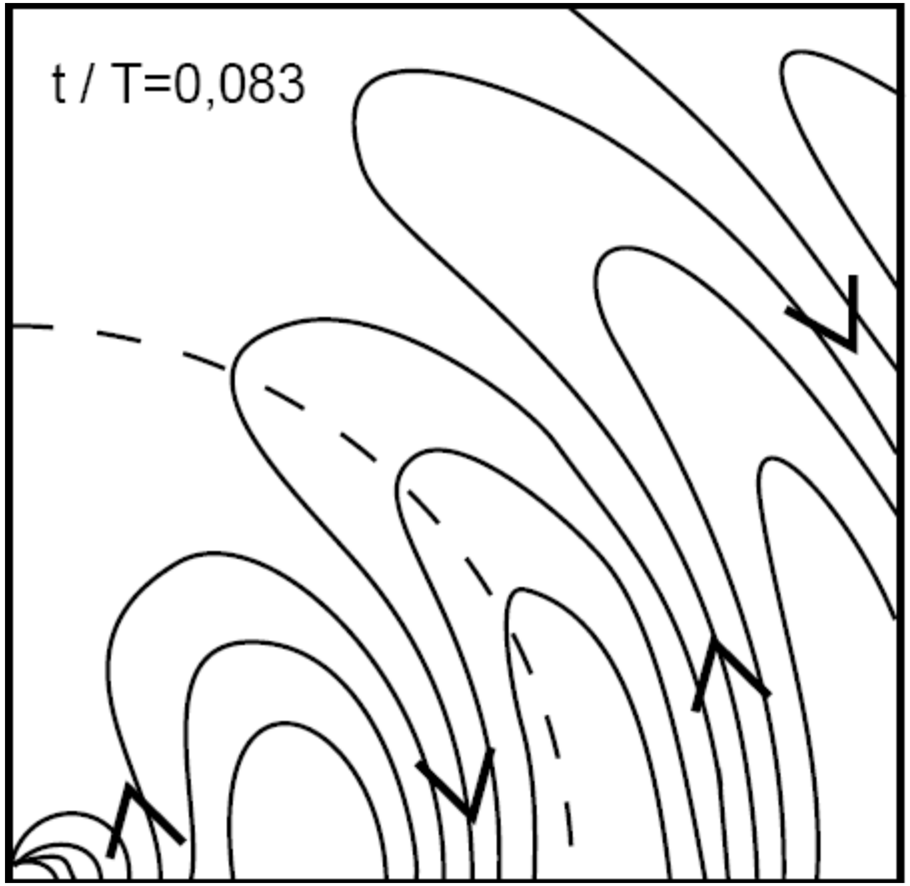


like a donut ..
(but without the hole in the center)

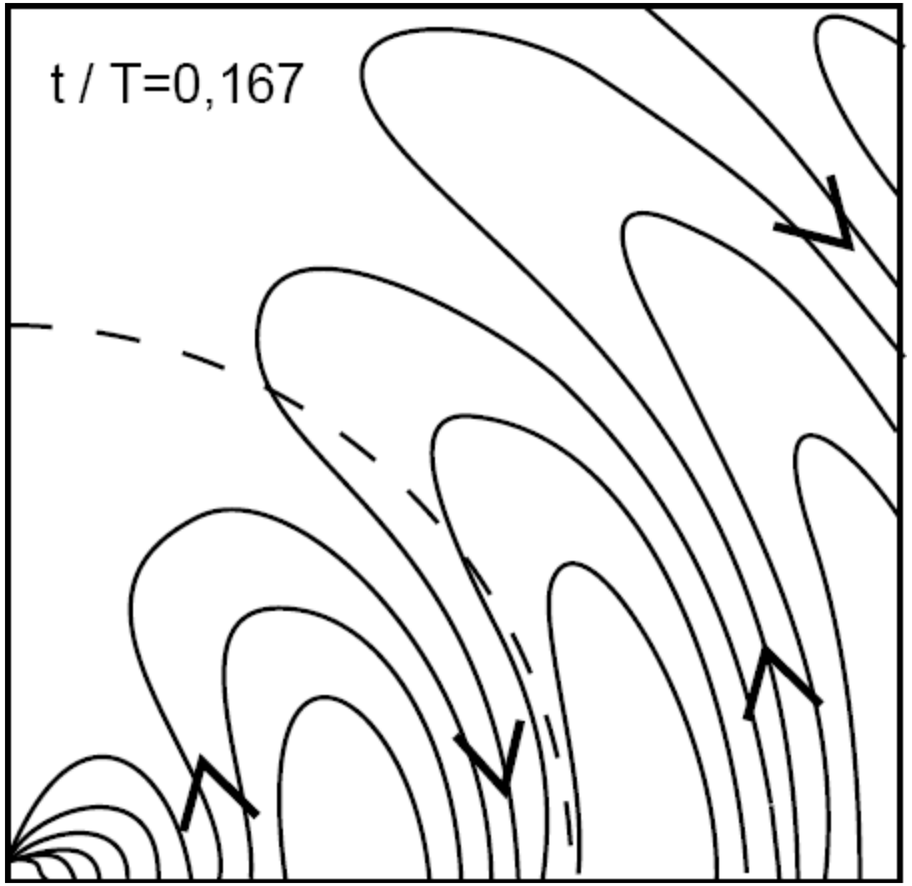
Electric Field of the Hertzian Dipole



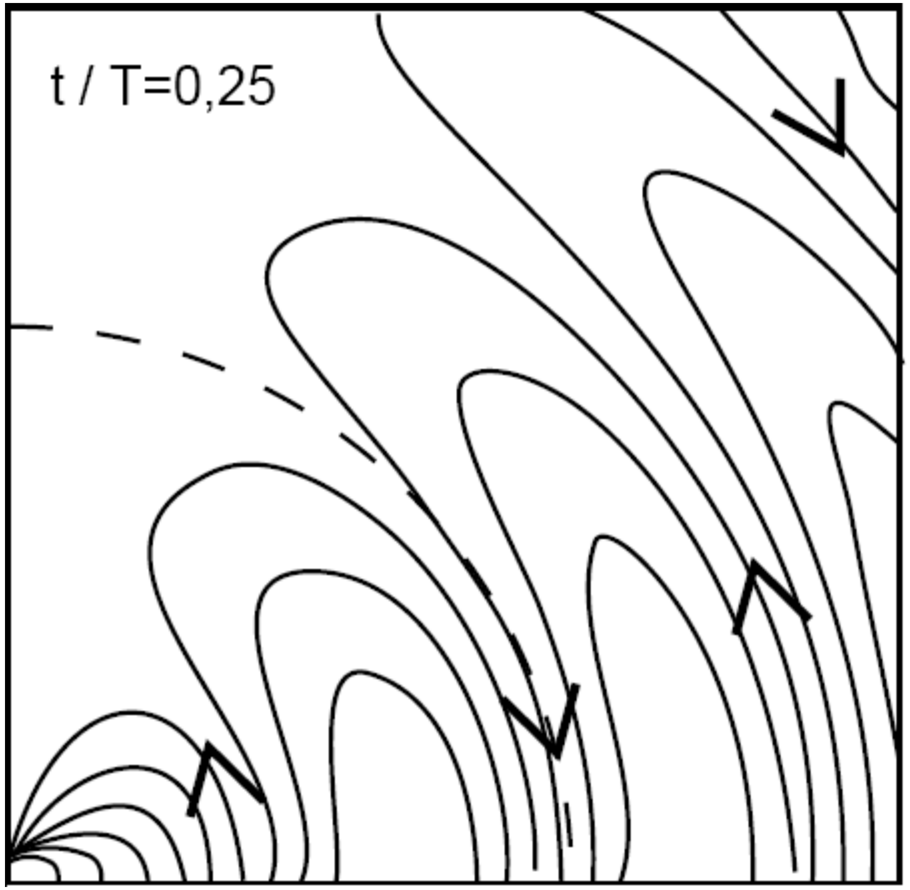
Electric Field of the Hertzian Dipole



Electric Field of the Hertzian Dipole



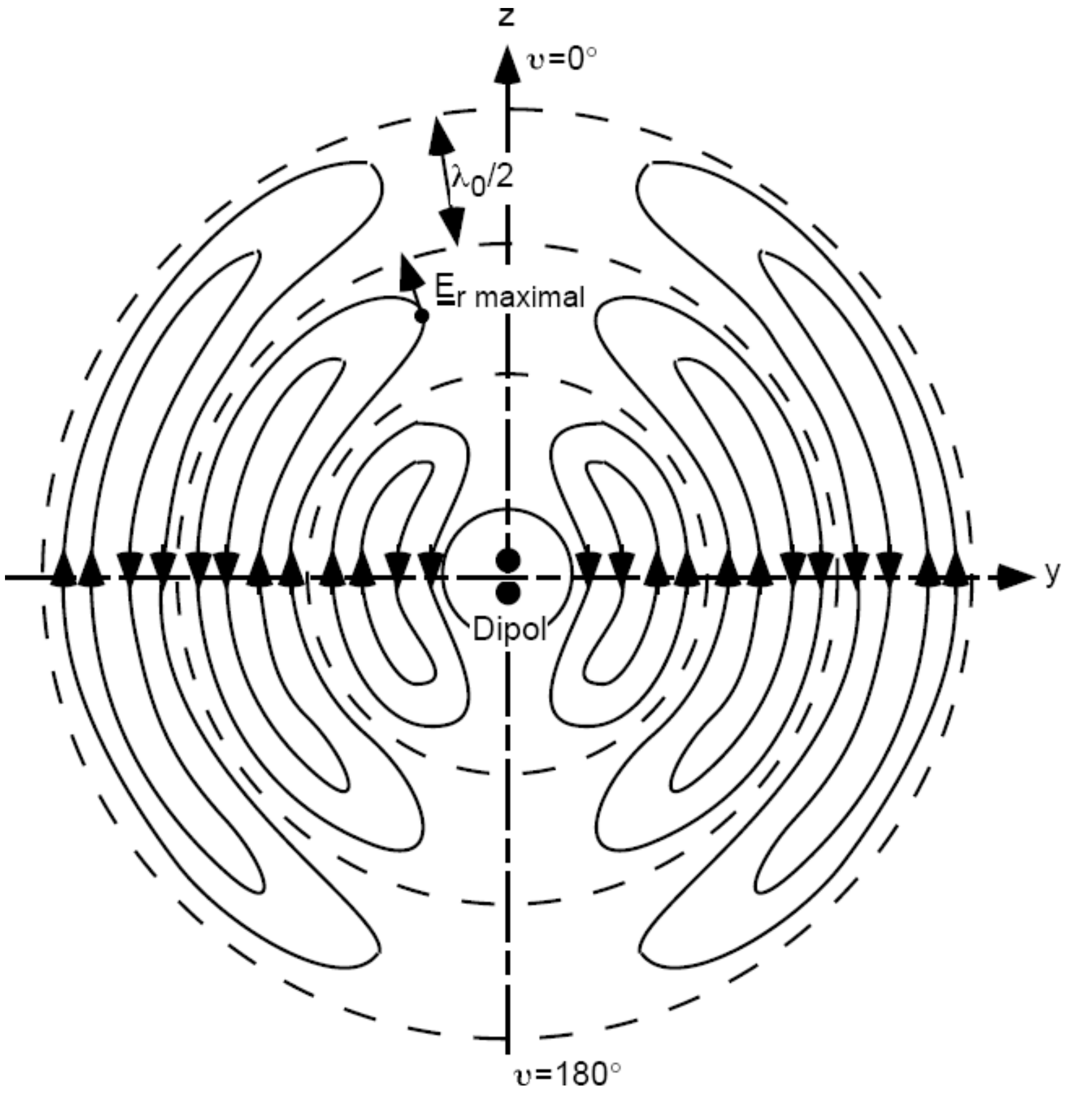
Electric Field of the Hertzian Dipole



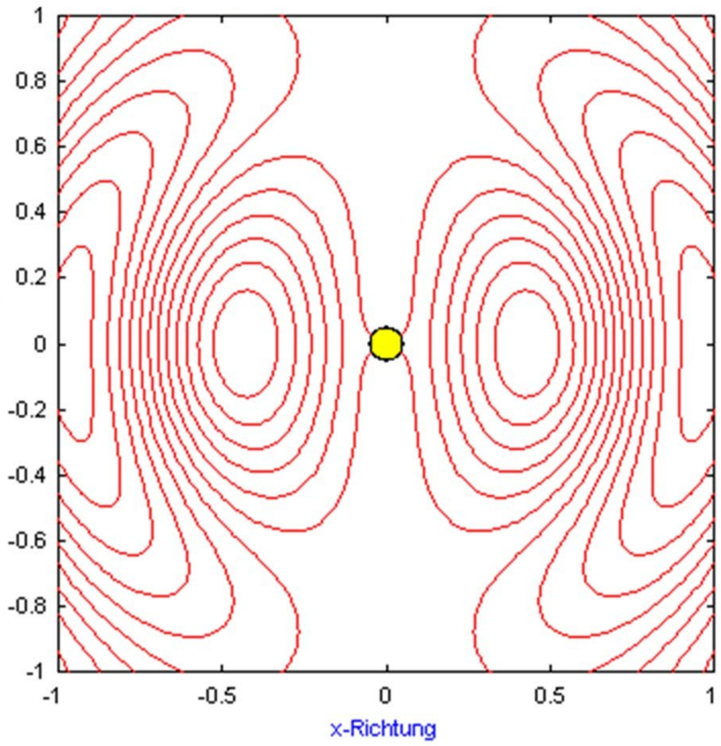
Electric Field of the Hertzian Dipole

Electric Field of the Hertzian Dipole

Electric Field of the Hertzian Dipole



Hertzian Dipole Radiation



Source: http://www.hs-weingarten.de/~kark/Forschung/index.htm

5.1 Problem 1

If only the far field is of interest, the curl operator

$$\operatorname{curl} \underline{\vec{A}} = \begin{pmatrix} \frac{1}{r\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\underline{A}_\varphi) - \frac{\partial \underline{A}_\vartheta}{\partial\varphi}\right) \\ \frac{1}{r\,\sin\vartheta}\frac{\partial \underline{A}_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r\,\underline{A}_\varphi) \\ \frac{1}{r}\frac{\partial}{\partial r}(r\,\underline{A}_\vartheta) - \frac{1}{r}\frac{\partial \underline{A}_r}{\partial\vartheta} \end{pmatrix}$$

can be simplified because of the known radial dependence of all field components.

a) Give the general radial dependence of all field components in a far field distance.

b) What terms in the equation above are the dominant terms?
Show how the curl operator can be simplified if the point of observation is in a far field distance.

c) With the above simplification, give all electric and magnetic field components in a far field distance.

$$\underline{H}_r = 0$$

$$\underline{H}_\vartheta = 0$$

$$\underline{H}_\varphi = j\frac{k\,I\,h}{4\pi}\cdot\frac{1}{r}\cdot\sin\vartheta\cdot e^{-jkr}$$

$$\underline{E}_r = 0$$

$$\underline{E}_\vartheta = j\frac{k\,I\,h}{4\pi}\cdot Z_F\cdot\frac{1}{r}\cdot\sin\vartheta\cdot e^{-jkr}$$

$$\underline{E}_\varphi = 0$$

a) The radial dependence in a far-field distance is

$$\frac{e^{-jkr}}{r}$$

with the wavenumber $k = \frac{2\pi}{\lambda}$. This holds for all field components and the vector field components $A_r, A_\vartheta, A_\varphi$.

5.1 Problem 1

If only the far field is of interest, the curl operator

$$\operatorname{curl}\vec{A} = \begin{pmatrix} \frac{1}{r\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}(\sin\vartheta\,A_\varphi) - \frac{\partial A_\vartheta}{\partial\varphi}\right) \\ \frac{1}{r\sin\vartheta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r\,A_\varphi) \\ \frac{1}{r}\frac{\partial}{\partial r}(r\,A_\vartheta) - \frac{1}{r}\frac{\partial A_r}{\partial\vartheta} \end{pmatrix}$$

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c) With the above simplification, give all electric and magnetic field components in a far field distance.

$\frac{1}{r^2}$

$\phi \times \frac{e^{-jkr}}{r}$ → (dominant) → Value ↑

b) For derivatives $\frac{\partial \ldots}{\partial\vartheta}, \frac{\partial \ldots}{\partial\varphi}$ the radial dependency will still be $\frac{e^{-jkr}}{r}$. These terms are multiplied with $\frac{1}{r}$ so that a $\frac{1}{r^2}$ dependency exists.

The terms $\frac{1}{r}\frac{\partial}{\partial r}(r A_{\vartheta,\varphi})$ will become

$$\frac{1}{r}\frac{\partial}{\partial r}(r A_{\vartheta,\varphi}) = \frac{1}{r}\frac{\partial}{\partial r}\left(e^{-jkr}\right) = -jk\frac{e^{-jkr}}{r} = -jk A_{\vartheta,\varphi}$$

giving a $\frac{1}{r}$ dependency. Therefore, these terms are dominant.

$$\operatorname{curl}\underline{\vec{A}} = \begin{pmatrix} 0 \\ jk\underline{A}_\varphi \\ -jk\underline{A}_\vartheta \end{pmatrix} = jk\begin{pmatrix} 0 \\ \underline{A}_\varphi \\ -\underline{A}_\vartheta \end{pmatrix}$$

$$\Rightarrow \vec{\underline{B}} = \operatorname{curl}\vec{\underline{A}}\,; \quad \begin{pmatrix} B_r \\ B_\vartheta \\ B_\varphi \end{pmatrix} = jk\begin{pmatrix} 0 \\ A_\varphi \\ -A_\vartheta \end{pmatrix}$$

$$\Rightarrow B_r \approx 0\,;\quad \underline{B}_\vartheta = jk\underline{A}_\varphi\,;\quad \underline{B}_\varphi = -jk\underline{A}_\vartheta$$

(in a far-field distance)

5.1 Problem 1

If only the far field is of interest, the curl operator

$$\operatorname{curl}\underline{\vec{A}} = \begin{pmatrix} \frac{1}{r\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\underline{A}_\varphi) - \frac{\partial \underline{A}_\vartheta}{\partial\varphi}\right) \\ \frac{1}{r\sin\vartheta}\frac{\partial \underline{A}_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r\,\underline{A}_\varphi) \\ \frac{1}{r}\frac{\partial}{\partial r}(r\,\underline{A}_\vartheta) - \frac{1}{r}\frac{\partial \underline{A}_r}{\partial\vartheta} \end{pmatrix}$$

can be simplified because of the known radial dependence of all field components.

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Show how the curl operator can be simplified if the point of observation is in a far field distance.

c) With the above simplification, give all electric and magnetic field components in a far field distance.

$$\begin{pmatrix} B_r \\ B_\vartheta \\ B_\varphi \end{pmatrix} = jk \begin{pmatrix} 0 \\ A_\varphi \\ -A_\vartheta \end{pmatrix}$$

c)

$$\underline{\vec{H}} = \frac{1}{\mu}\underline{\vec{B}} = \frac{jk}{\mu}\begin{pmatrix} 0 \\ A_\varphi \\ -A_\vartheta \end{pmatrix} = j\frac{\omega\sqrt{\mu\varepsilon}}{\mu}\begin{pmatrix} 0 \\ A_\varphi \\ -A_\vartheta \end{pmatrix}$$

$$= j\omega\sqrt{\frac{\varepsilon}{\mu}}\begin{pmatrix} 0 \\ A_\varphi \\ -A_\vartheta \end{pmatrix} = j\frac{\omega}{Z_F}\begin{pmatrix} 0 \\ A_\varphi \\ -A_\vartheta \end{pmatrix} = \begin{pmatrix} H_r \\ H_\vartheta \\ H_\varphi \end{pmatrix}$$

$\vec{E}$-Field: $\operatorname{curl}\vec{H} = \frac{\partial\vec{D}}{\partial t} \Rightarrow \operatorname{curl}\underline{\vec{H}} = j\omega\varepsilon\,\underline{\vec{E}}$

$$\Rightarrow \underline{\vec{E}} = \frac{1}{j\omega\varepsilon}\operatorname{curl}\underline{\vec{H}}$$

$$\operatorname{curl}\underline{\vec{H}} \approx jk\begin{pmatrix} 0 \\ H_\varphi \\ -H_\vartheta \end{pmatrix}$$

$$\underline{\vec{E}} = \frac{k}{\omega\varepsilon}\begin{pmatrix} 0 \\ H_\varphi \\ -H_\vartheta \end{pmatrix} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\varepsilon}\begin{pmatrix} 0 \\ H_\varphi \\ -H_\vartheta \end{pmatrix} = Z_F\begin{pmatrix} 0 \\ H_\varphi \\ -H_\vartheta \end{pmatrix}$$

$$= j\omega\begin{pmatrix} 0 \\ -A_\vartheta \\ -A_\varphi \end{pmatrix};$$

$E_r \approx 0$; $E_\vartheta = -j\omega A_\vartheta$; $E_\varphi = -j\omega A_\varphi$

$$\left(\vec{E} = -\frac{\partial\vec{A}}{\partial t}\right)$$

(in a far field distance!) (But $E_r = 0$!)

5.2 Problem 2

A Hertzian dipole with a current magnitude I is oriented in x-direction.

a) Determine the magnetic vector potential $\vec{A}$.

b) Determine the magnetic vector field $\vec{H}$ in a far field distance.

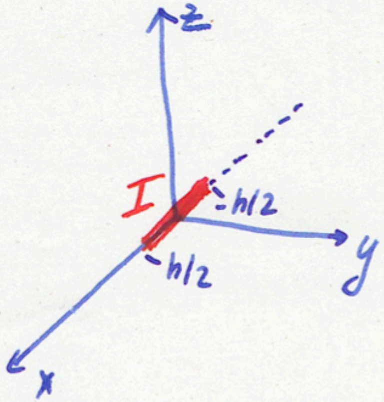
c) Determine the electric vector field $\vec{E}$ in a far field distance.

d) Give the radiation characteristic in a far field distance and sketch it in all three main planes.

$$\underline{\vec{A}}(P) = \frac{\mu}{4\pi} \iiint_V \frac{\underline{\vec{J}}(P') \cdot \mathrm{e}^{-jkR_{PP'}}}{R_{PP'}} dV'$$

$z = r \cos\theta$

a) $$\underline{\vec{A}}(P) = \frac{\mu}{4\pi} \iiint_V \frac{\vec{J}(P')\, e^{-jkR_{PP'}}}{R_{PP'}} dV'$$



Hertzian Dipole: $R_{PP'} \approx r$

$\vec{J}(P') = I\, dx\, \vec{e}_x$

$$\Rightarrow \underline{\vec{A}}(P) = \frac{\mu}{4\pi} \int_{-h/2}^{h/2} I \frac{e^{-jkr}}{r} dx\, \vec{e}_x$$

$$= \frac{\mu I h}{4\pi} \cdot \frac{e^{-jkr}}{r} \vec{e}_x$$

Cartesian coordinates → spherical coordinates:

$x = r \sin\vartheta \cos\varphi \quad \Big|\ \text{grad} \ldots$

$$\text{grad}\, x = \frac{\partial x}{\partial r} \vec{e}_r + \frac{1}{r} \cdot \frac{\partial x}{\partial \vartheta} \vec{e}_\vartheta + \frac{1}{r \sin\vartheta} \cdot \frac{\partial x}{\partial \varphi} \vec{e}_\varphi$$

$$\vec{e}_x = \sin\vartheta \cos\varphi\, \vec{e}_r + \cos\vartheta \cos\varphi\, \vec{e}_\vartheta - \sin\varphi\, \vec{e}_\varphi$$

$$\Rightarrow \underline{\vec{A}}(P) = \frac{\mu I h}{4\pi} \cdot \frac{e^{-jkr}}{r} \cdot \begin{pmatrix} \sin\vartheta \cos\varphi \\ \cos\vartheta \cos\varphi \\ -\sin\varphi \end{pmatrix} = \begin{pmatrix} A_r \\ A_\vartheta \\ A_\varphi \end{pmatrix}$$

5.2 Problem 2

A Hertzian dipole with a current magnitude I is oriented in x-direction.

a) Determine the magnetic vector potential $\vec{A}$.

b) Determine the magnetic vector field $\vec{H}$ in a far field distance.

c) Determine the electric vector field $\vec{E}$ in a far field distance.

d) Give the radiation characteristic in a far field distance and sketch it in all three main planes.

$$\operatorname{curl}\vec{A} = \begin{pmatrix} 0 \\ jkA_\varphi \\ -jkA_\vartheta \end{pmatrix} = jk\begin{pmatrix} 0 \\ A_\varphi \\ -A_\vartheta \end{pmatrix}$$

$$\vec{A}(P) = \frac{\mu I h}{4\pi}\cdot\frac{e^{-jkr}}{r}\cdot\begin{pmatrix} \sin\vartheta\cos\varphi \\ \cos\vartheta\cos\varphi \\ -\sin\varphi \end{pmatrix} = \begin{pmatrix} A_r \\ A_\vartheta \\ A_\varphi \end{pmatrix}$$

$$E_r \approx 0\ ; \quad E_\vartheta = -j\omega A_\vartheta\ ; \quad E_\varphi \approx -j\omega A_\varphi$$

Formula

b) $\vec{H} = \frac{1}{\mu}\operatorname{curl}\vec{A}$; Far Field: $\operatorname{curl}\vec{A} \approx jk\begin{pmatrix} 0 \\ A_\varphi \\ -A_\vartheta \end{pmatrix}$

$$\vec{H} \approx -jk\frac{Ih}{4\pi}\cdot\frac{e^{-jkr}}{r}\cdot\begin{pmatrix} 0 \\ \sin\varphi \\ \cos\vartheta\cos\varphi \end{pmatrix}$$

c) $\vec{E} \approx -j\omega\begin{pmatrix} 0 \\ A_\vartheta \\ A_\varphi \end{pmatrix}$ with $\operatorname{curl}\vec{H} \approx jk\begin{pmatrix} 0 \\ H_\varphi \\ -H_\vartheta \end{pmatrix}$

$$= -j\omega\mu\frac{Ih}{4\pi}\cdot\frac{e^{-jkr}}{r}\begin{pmatrix} 0 \\ \cos\vartheta\cos\varphi \\ -\sin\varphi \end{pmatrix}$$

$$\left(\frac{|\vec{E}|}{|\vec{H}|} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = Z_F\right)$$

5.2 Problem 2

A Hertzian dipole with a current magnitude I is oriented in x-direction.

a) Determine the magnetic vector potential $\vec{A}$.

b) Determine the magnetic vector field $\vec{H}$ in a far field distance.

c) Determine the electric vector field $\vec{E}$ in a far field distance.

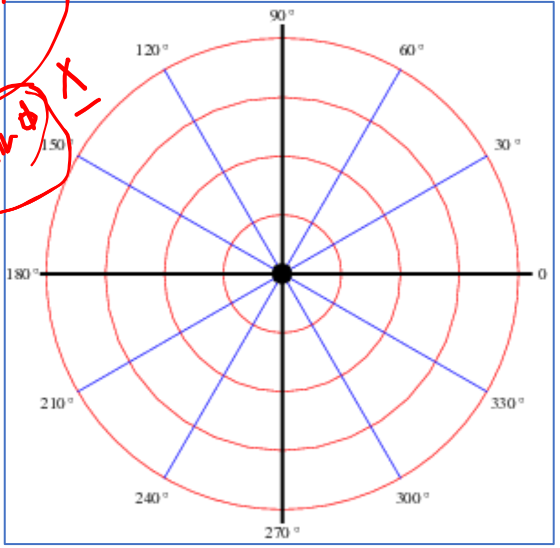
d) Give the radiation characteristic in a far field distance and sketch it in all three main planes.

$$\vec{E} \approx -j\omega \begin{pmatrix} 0 \\ A_\vartheta \\ A_\varphi \end{pmatrix} \quad \text{with } \operatorname{curl} \vec{H} \approx jk \begin{pmatrix} 0 \\ H_\varphi \\ -H_\vartheta \end{pmatrix}$$

$$= -j\omega\mu \frac{Ih}{4\pi} \cdot \frac{e^{-jkr}}{r} \begin{pmatrix} 0 \\ \cos\vartheta \cos\varphi \\ -\sin\varphi \end{pmatrix}$$

$E_\theta = \cos\theta \cos\phi$

$E_\phi = (-\sin\phi)$

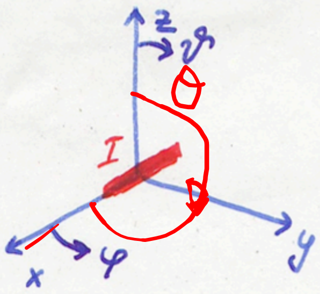


$|\vec{E}_\theta + \vec{E}_\phi|$

d) Radiation characteristic:

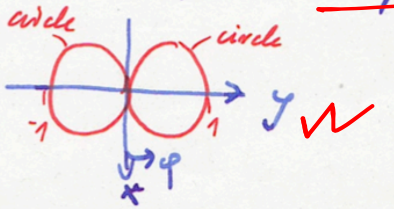
$$C(\vartheta,\varphi) = \frac{|\vec{E}(\vartheta,\varphi)|}{|\vec{E}|_{max}} = \frac{|\vec{H}(\vartheta,\varphi)|}{|\vec{H}|_{max}}$$

$$= \sqrt{\cos^2\vartheta \cos^2\varphi + \sin^2\varphi}$$

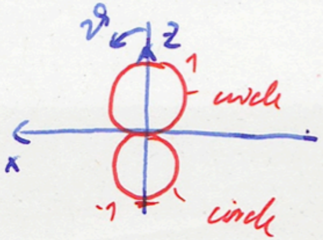


Three main planes:
- xy plane ($\vartheta = 90°$)
- xz plane ($\varphi = 0°$)
- yz plane ($\varphi = 90°$)

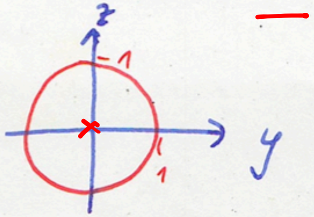
xy plane: $C(\vartheta,\varphi) = \sin\varphi$



xz-plane: $C(\vartheta,\varphi) = \cos\vartheta$



yz-plane: $C(\vartheta,\varphi) = 1$



5.3 Problem 3

A dipole is put into the origin of a Cartesian coordinate system. The dipole has a height h and carries a harmonically varying electric current of amplitude I. The frequency is $f = 1\,\mathrm{GHz}$ and the so-called current moment $I \cdot h$ is $I \cdot h = 1\mathrm{Am}$.

The orientation of the dipole is $\vec{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

A second dipole (of same dimensions) is used as a receive antenna. It is placed on the z-axis in a far-field distance of $r = 100\,\mathrm{m}$.

a) What is the polarization of the transmitted wave?

b) Calculate the electric and magnetic field strengths' amplitudes at the position of the receive antenna.

c) Calculate the mean power flow density at the position of the receive antenna.

a) Linear polarization along the x-axis.

$$\underline{H}_r = 0$$

$$\underline{H}_\vartheta = 0$$

$$\underline{H}_\varphi = j\frac{k\,I\,h}{4\pi} \cdot \frac{1}{r} \cdot \sin\vartheta \cdot \mathrm{e}^{-jkr}$$

$$\underline{E}_r = 0$$

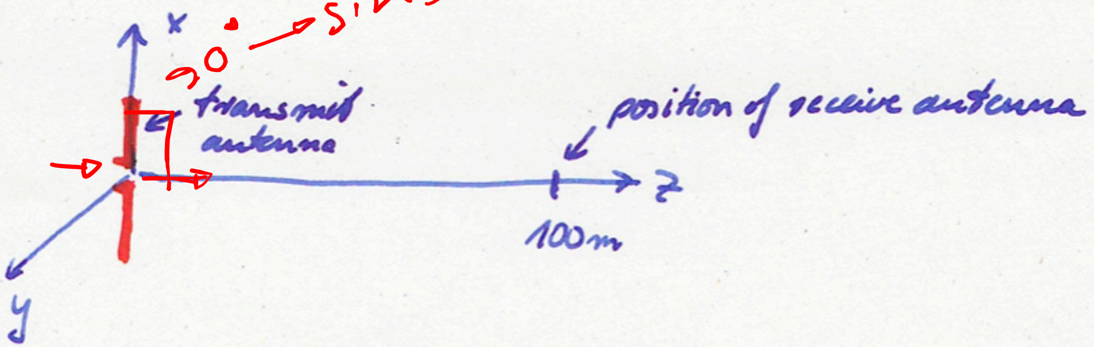
$$\underline{E}_\vartheta = j\frac{k\,I\,h}{4\pi} \cdot Z_F \cdot \frac{1}{r} \cdot \sin\vartheta \cdot \mathrm{e}^{-jkr}$$

$$\underline{E}_\varphi = 0$$

e^{-600}

b)

$90° \rightarrow \sin 90° = 1$



$$|\vec{\underline{H}}| = k\frac{Ih}{4\pi} \cdot \frac{1}{r} = \frac{2\pi}{\lambda} \cdot \frac{Ih}{4\pi} \cdot \frac{1}{r} = \frac{Ih}{2\lambda} \cdot \frac{1}{r}$$

$$f = 1\,\mathrm{GHz} \Rightarrow \lambda = 0.3\,\mathrm{m}$$

$$\Rightarrow |\vec{\underline{H}}| = \frac{1\,\mathrm{Am}}{0.6\,\mathrm{m}} \cdot \frac{1}{100\,\mathrm{m}} = \frac{1}{60}\frac{\mathrm{A}}{\mathrm{m}} \approx 16.67\,\frac{\mathrm{mA}}{\mathrm{m}}$$

$$|\vec{\underline{E}}| = Z_F \cdot |\vec{\underline{H}}| = 120\pi\,\Omega \cdot 0.01667\,\frac{\mathrm{A}}{\mathrm{m}} = 6.284\,\frac{\mathrm{V}}{\mathrm{m}}$$

5.3 Problem 3

A dipole is put into the origin of a Cartesian coordinate system. The dipole has a height h and carries a harmonically varying electric current of amplitude I. The frequency is $f = 1\,\text{GHz}$ and the so-called current moment $I \cdot h$ is $I \cdot h = 1\text{Am}$.

The orientation of the dipole is $\vec{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

A second dipole (of same dimensions) is used as a receive antenna. It is placed on the z-axis in a far-field distance of $r = 100\,\text{m}$.

a) What is the polarization of the transmitted wave?

b) Calculate the electric and magnetic field strengths' amplitudes at the position of the receive antenna.

c) Calculate the mean power flow density at the position of the receive antenna.

H_ϕ, E_θ

$$\vec{S} = \frac{1}{2} Z_F \cdot \left(\frac{k\,I\,h}{4\pi}\right)^2 \cdot \frac{\sin^2\vartheta}{r^2}\,\vec{e}_r$$

c) $$|\vec{S}| = \frac{1}{2} Z_F \left(\frac{k\,I h}{4\pi}\right)^2 \frac{1}{r^2} = \frac{1}{2} Z_F \left(\frac{I h}{2\lambda}\right)^2 \frac{1}{r^2}$$

$$= \frac{1}{2} \cdot 120\pi\,\Omega \left(\frac{1\,\text{Am}}{2 \cdot 0.3\,\text{m}}\right)^2 \cdot \frac{1}{(100\,\text{m})^2} = 0.0524\,\frac{\text{W}}{\text{m}^2}$$

$$\left(= \frac{1}{2} |\vec{E}| \cdot |\vec{H}|\right)$$

5.3 Problem 3

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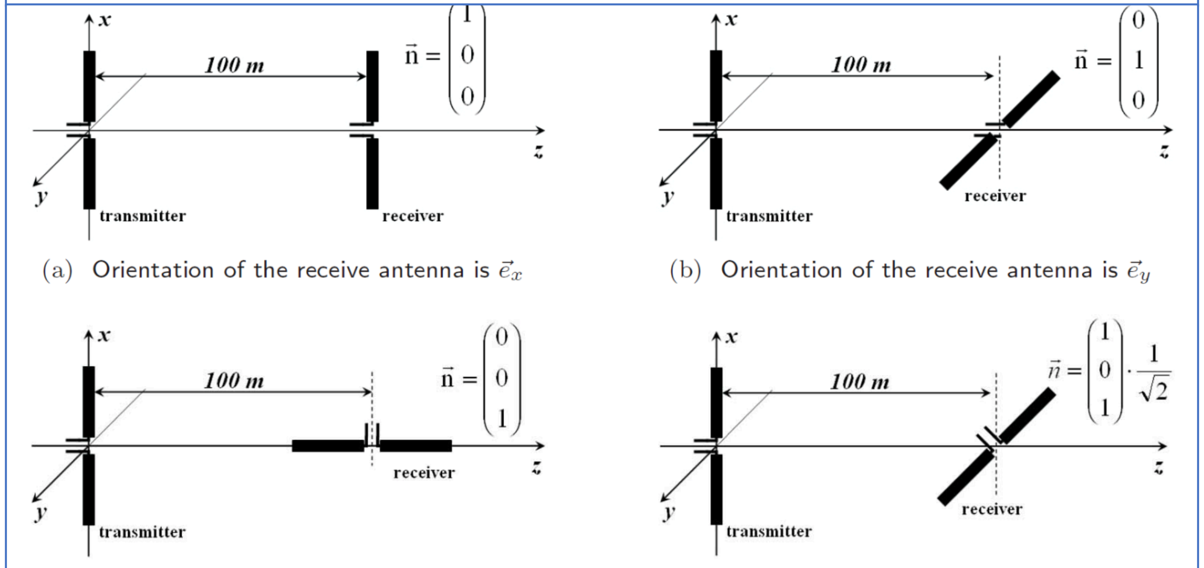

(a) Orientation of the receive antenna is $\vec{e}_x$

(b) Orientation of the receive antenna is $\vec{e}_y$

(c) Orientation of the receive antenna is $\vec{e}_z$

(d) Orientation of the receive antenna is $\frac{\vec{e}_x + \vec{e}_z}{\sqrt{2}}$

Figs. a) to d) show different possible orientations of the receive antenna.

d) Regarding the received power, what is the optimal orientation of the receive antenna?

e) How much power is received by the other three combinations compared to the optimal one of task d)?

d) Fig a) shows the optimal orientation of the receive antenna.

e) in Fig b) the receive antenna is perpendicularly oriented to the transmit antenna ⇒ no reception; the same in Fig. c) in Fig d) the induced electric field is $\frac{1}{\sqrt{2}}$ of the maximum available voltage; power is reduced by factor of 2. (−3dB).

$20\log_{10}(1/\sqrt{2})$